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CONVERGENCE OF DIRICHLET FORMS AND BESOV NORMS ON SCALE IRREGULAR SIERPIŃSKI GASKETS

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Abstract

In this paper, we construct equivalent semi-norms of local and non-local Dirichlet forms on scale irregular Sierpiński gaskets. These fractals are not necessarily self-similar, and have volume doubling Hausdorff measures which are not necessarily Ahlfors regular. We obtain that a sequence of non-local Dirichlet forms converges to a local Dirichlet form, which extends a convergence theorem of Bourgain, Brezis and Mironescu to the scale irregular Sierpiński gaskets for $p = 2$.

Keywords: Convergence; Dirichlet Form; Random Fractals; Doubling Measure.

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1. INTRODUCTION

Bourgain *et al.* proposed the limiting behavior of the norms of fractional Sobolev spaces on $\mathbb{R}^n$ ((see Theorem 2 in Ref. 1) with a suitable choice of ρ_n and $p = 2$)

$$\lim_{\beta\uparrow 2}(2-\beta)\int_D\int_D \frac{|u(x)-u(y)|^2}{|x-y|^{n+\beta}}dxdy = C_n\int_D |\nabla u(x)|^2dx, \tag{1}$$

for all $u \in W^{1,2}(D)$, where D is a smooth bounded domain in $\mathbb{R}^n$, and C_n is a positive constant depending only on n. The left-hand side is a sequence of non-local Dirichlet forms, which converges to the right-hand side, a local Dirichlet form on D. There are some papers devoted to generalize this limiting embedding theorem to fractional Sobolev spaces on fractals then. Using the sub-Gaussian estimates for the heat kernel of the local Dirichlet form $\mathcal{E}_{\text{loc}}$, Pietruska–Pałuba proved that the Dirichlet form $\mathcal{E}_\beta$ which is obtained from subordination of order β/β^* have the properties $(\beta^*-\beta)\mathcal{E}_\beta \to \mathcal{E}_{\text{loc}}$ as $\beta \uparrow \beta^*$ in Ref. 2. Later, Yang studied this problem and also studied the Mosco-convergence of Dirichlet forms on the Sierpiński gasket in Ref. 3. Grigor'yan and Yang further proved a discrete analog of (1) on the Sierpiński carpet in Ref. 4, using the Γ-convergence of the $B^{\beta}_{2,2}$-norms to the $B^{\beta^*}_{2,\infty}$-norm as $\beta \uparrow \beta^*$. After that, Gu and Lau studied the convergence of the $B^{\beta}_{2,2}$-norms to the $B^{\beta^*}_{2,\infty}$-norm and its associated Dirichlet forms on homogeneous p.c.f. self-similar sets in Refs. 5 and 6.

In this paper, we consider the convergence of Dirichlet forms on *scale irregular Sierpiński gaskets*, which are also called homogeneous random Sierpiński gaskets. Such fractals are a kind of Moran sets where many works have been developed to discuss the geometric measure properties (see for example Refs. 7–9), and they are also a class of V-variable fractals considered in Refs. 10–12, etc. The associated diffusion process on scale irregular Sierpiński gaskets has been extensively studied in Refs. 12–14. Other analytical aspects, such as spectral decimation, Dirichlet form and heat kernel estimate, were discussed, for example, in Refs. 15–17.

We follow the definitions in Refs. 17 and 18. For each $l \in \mathbb{N}\backslash\{1\}$, the Sierpiński-type gasket SG(l) is a self-similar set formed recursively by dividing the sides of each equilateral triangle in l, joining these points and removing all the smaller downward pointing triangles. Precisely, for a unit equilateral triangle (with side length 1) denoted by $\triangle \subset \mathbb{R}^2$, let V_0 be the set of its vertices $\{q_k \,|\, k \in \{0,1,2\}\}$. For each $i = (i_1, i_2) \in S_l$ where $S_l := \{(i_1,i_2) \in (\mathbb{N}\cup\{0\})^2 | i_1 + i_2 \le l-1\}$, we set $q^l_i := q_0 + \sum_{k=1}^2 (i_k/l)(q_k - q_0)$, then SG($l$) is the attractor of the IFS formed by all the contracting similarities $F^l_i(x) := q^l_i + l^{-1}(x - q_0), x \in \mathbb{R}^2$. According to the related definition of diffusion process on these sets in Sec. 2 of Ref. 18, the scale factors for SG(l) are length l, mass $m_l := l(l+1)/2$, (diffusion) time t_l, Hausdorff dimension

$$\alpha_l = \frac{\log m_l}{\log l}, \tag{2}$$

and walk dimension

$$\beta_l = \frac{\log t_l}{\log l}. \tag{3}$$

Let $K^{\boldsymbol{l}}$ denote the *level-$\boldsymbol{l}$ scale irregular Sierpiński gasket SG($\boldsymbol{l}$)* with an infinite sequence $\boldsymbol{l} = (l_n)_{n=1}^\infty$, where each l_n only takes two different values a and b in $\mathbb{N}\backslash\{1\}$. We set $W^{\boldsymbol{l}}_n := \prod_{k=1}^n S_{l_k}$ for each $n \in \mathbb{N}$ and $F^{\boldsymbol{l}}_w = F^{l_1}_{w_1} \circ \cdots \circ F^{l_n}_{w_n}$ for each $n \in \mathbb{N}$ and $w = w_1 \ldots w_n \in W^{\boldsymbol{l}}_n$. We define $K^{\boldsymbol{l}}$ to be the non-empty subset of $\triangle$ given by

$$K^{\boldsymbol{l}} := \bigcap_{n=1}^\infty \bigcup_{w\in W^{\boldsymbol{l}}_n} F^{\boldsymbol{l}}_w(\triangle).$$

Note that $\{\bigcup_{w\in W^{\boldsymbol{l}}_n} F^{\boldsymbol{l}}_w(\triangle)\}_{n=1}^\infty$ is a strictly decreasing sequence of non-empty compact subsets of $\triangle$, and that the intersection of the *level-n cells* satisfy that

$$F^{\boldsymbol{l}}_w(\triangle) \cap F^{\boldsymbol{l}}_v(\triangle) = F^{\boldsymbol{l}}_w(V_0) \cap F^{\boldsymbol{l}}_v(V_0),$$

for any $n \in \mathbb{N}$ and any $w, v \in W^{\boldsymbol{l}}_n$ with $w \neq v$. We also set $V^{\boldsymbol{l}}_0 := V_0$ and $V^{\boldsymbol{l}}_n := \bigcup_{|w|=n} F^{\boldsymbol{l}}_w(V_0)$ for any $n \in \mathbb{N}$, where $|w|$ denotes the length of word w, so that $\{V^{\boldsymbol{l}}_n\}_{n=0}^\infty$ is a strictly increasing sequence of finite subsets of $K^{\boldsymbol{l}}$ and $\bigcup_{n=0}^\infty V^{\boldsymbol{l}}_n$ is dense in $K^{\boldsymbol{l}}$ (under the Euclidean metric). Such construction appeared in Refs. 13 and 18 (Sec. 5 of Ref. 17 and Sec. 23 of Ref. 19), and an illustration is given in Fig. 1.

By Sec. 2 of Ref. 18, a diffusion process can be constructed on the fractal $K^{\boldsymbol{l}}$ built from any sequence $\boldsymbol{l}$, but for the critical exponents to be well defined we will assume throughout that the sequence is stationary and ergodic. Precisely, let $[n]_a$, $[n]_b$ be the number of occurrence of a and b in $\boldsymbol{l}|_n$, the first n digits of $\boldsymbol{l}$, respectively, then we only consider those $\boldsymbol{l}$ for which $\lim_{n\to\infty} \frac{[n]_a}{[n]_b}$ exists. Then by

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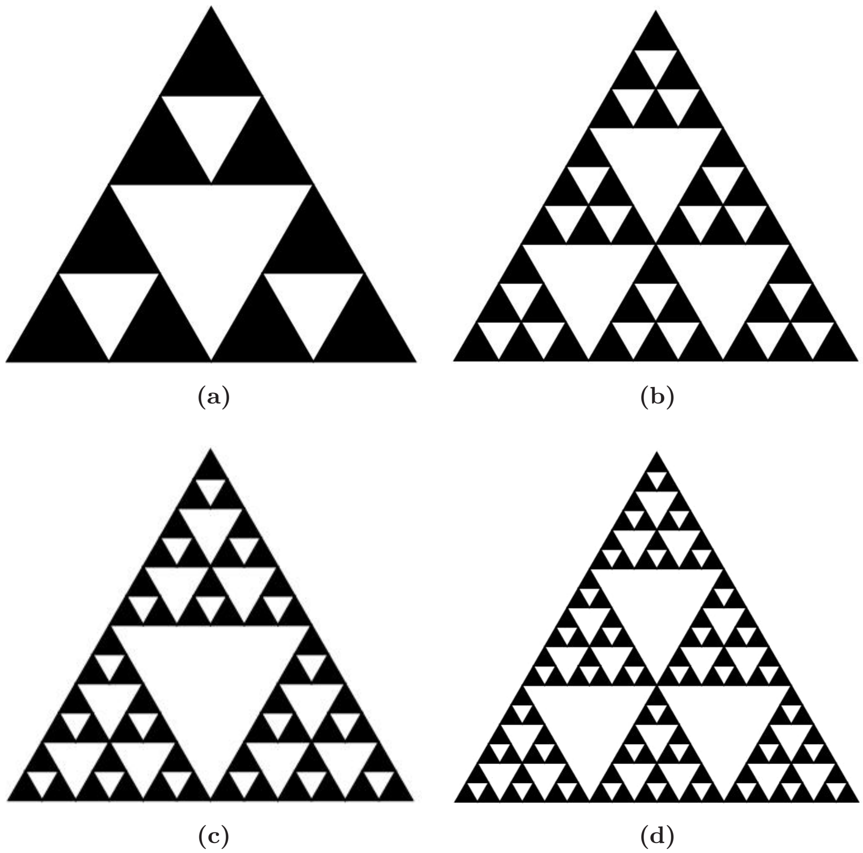

Fig. 1 The level-$\boldsymbol{l}$ scale irregular Sierpiński gasket SG($\boldsymbol{l}$). **(a)** SG(2) with mass $m_a = 3$, length $a = 2$ and time $t_a = 5$. **(b)** SG(3) with mass $m_b = 6$, length $b = 3$, and time $t_b = 90/7$ (see for example p. 2 of Ref. 12). **(c)** SG($\boldsymbol{l}$) with $\rho_n = 2^{-[n]_a}3^{-[n]_b}$, where $\rho_1 = 2^{-1}$, $\rho_2 = 2^{-1}3^{-1}$, $\rho_3 = 2^{-2}3^{-1}$. **(d)** SG($\boldsymbol{l}$) with $\rho_n = 2^{-[n]_a}3^{-[n]_b}$, where $\rho_1 = 3^{-1}$, $\rho_2 = 3^{-2}$, $\rho_3 = 2^{-1}3^{-2}$.

Lemma 2.1 and Theorem 2.2 of Ref. 18, the Hausdorff dimension α of $K^{\boldsymbol{l}}$ is given by

$$\alpha = \lim_{n\to\infty} \frac{[n]_a \log m_a + [n]_b \log m_b}{[n]_a \log a + [n]_b \log b}.$$

For simplicity, we will regard the empty word $\emptyset$ to be the unique word of length 0, and the corresponding notions are naturally understood to be $F^{\boldsymbol{l}}_\emptyset := \mathrm{Id}, K^{\boldsymbol{l}}_\emptyset := K, V^{\boldsymbol{l}}_\emptyset := V_0$. We also make the convention that $[0]_a = [0]_b = 0$.

Let $\rho_n = a^{-[n]_a}b^{-[n]_b}$ ($n \geq 0$) be the diameter of level-n cells. Define the volume function

$$\psi(r) = \psi(\rho_n) := m_a^{-[n]_a} m_b^{-[n]_b} \quad \text{for}$$
$$\rho_{n+1} < r \leq \rho_n (n \geq 0),$$
$$\psi(r) = \psi(1) = 1 \quad \text{for } r \geq 1. \tag{4}$$

It follows that

$$\psi(r) \asymp r^{\alpha(n)} \quad (\rho_{n+1} < r \leq \rho_n), \tag{5}$$

where $\alpha(n) = \frac{[n]_a \log m_a + [n]_b \log m_b}{[n]_a \log a + [n]_b \log b}$.

In this paper, we consider the restriction of the Euclidean metric d on $\mathbb{R}^2$ to $K^{\boldsymbol{l}}$, which is bi-Lipschitz equivalent to a geodesic metric $d_{\boldsymbol{l}}$ on $K^{\boldsymbol{l}}$ as discussed in Lemma 2.4 of Ref. 13. The standard measure-theoretic arguments immediately show that there exists a unique Borel probability measure μ on $K^{\boldsymbol{l}}$ such that

$$\mu(K^{\boldsymbol{l}}_w) = \#(W^{\boldsymbol{l}}_m)^{-1}, \quad \text{for any } w \in W^{\boldsymbol{l}}_m, \tag{6}$$

then we also have $\#(W^{\boldsymbol{l}}_m)^{-1} = \psi(\rho_m)$ by (4). Define $|x-y| := d(x,y)$. Corollary 2.2 says that the measure of any metric ball $B(x,r) := \{y \in K^{\boldsymbol{l}} : |x-y| < r\}$ with $0 < r \leq 1$ satisfies

$$\mu(B(x,r)) \asymp \psi(r). \tag{7}$$

This measure is natural, since that, if the Hausdorff measure $\mathcal{H}^\alpha$ exists, that is, $0 < \mathcal{H}^\alpha(K) < \infty$ (see a criterion in Theorem 1.1(i) of Ref. 8), then it is equivalent to μ. This is because $\mathcal{H}^\alpha(K_w) = \mathcal{H}^\alpha(K)\mu(K_w)$ as level-k cells are only translations of each other for all $w \in W_k, k \geq 1$, and any measurable set of K can be approximated by some cells of level k (letting $k \to \infty$), which means that $\mathcal{H}^\alpha = \mathcal{H}^\alpha(K)\mu$.

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By Sec. 2, Eq. (2.3) of Ref. 18, the walk dimension β^* for the diffusion on $K^{\boldsymbol{l}}$ is given by

$$\beta^* = \lim_{n\to\infty} \frac{[n]_a \log t_a + [n]_b \log t_b}{[n]_a \log a + [n]_b \log b}. \tag{8}$$

Similarly, define the function

$$\begin{aligned}&\varphi(r) = \varphi(\rho_n) := t_a^{-[n]_a} t_b^{-[n]_b} \quad \text{for}\\ &\rho_{n+1} < r \le \rho_n \quad (n \ge 0),\\ &\varphi(r) = \varphi(1) = 1 \quad \text{for } r \ge 1.\end{aligned} \tag{9}$$

Therefore,

$$\varphi(r) \asymp r^{\beta(n)} \quad (\rho_{n+1} < r \le \rho_n), \tag{10}$$

where $\beta(n) = \frac{[n]_a \log t_a + [n]_b \log t_b}{[n]_a \log a + [n]_b \log b}$. We will see that $\psi(r), \varphi(r)$ satisfy a scaling property in Lemma 2.1.

Next, for any $\beta \in (\alpha, \beta^*)$, define semi-norms $[u]_{B^\beta_{2,\infty}}$ and $[u]_{B^\beta_{2,2}}$ for $u \in L^2(K^{\boldsymbol{l}}, \mu)$ by

$$[u]^2_{B^\beta_{2,\infty}} := \sup_{r>0} \varphi(r)^{-\frac{\beta}{\beta^*}} \psi(r)^{-1} \times \int_{K^{\boldsymbol{l}}} \int_{B(x,r)} |u(x) - u(y)|^2 d\mu(y) d\mu(x), \tag{11}$$

$$[u]^2_{B^\beta_{2,2}} := \int_0^1 \left(\int_{K^{\boldsymbol{l}}} \int_{B(x,r)} \varphi(r)^{-\frac{\beta}{\beta^*}} \psi(r)^{-1} \times |u(x) - u(y)|^2 d\mu(y) d\mu(x) \right) \frac{dr}{r}. \tag{12}$$

Define $B^\beta_{2,\infty} := \{u \in L^2(K^{\boldsymbol{l}}, \mu) : \|u\|_{B^\beta_{2,\infty}} < \infty\}$ with the norm $\|u\|_{B^\beta_{2,\infty}} := \|u\|_{L^2(K^{\boldsymbol{l}},\mu)} + [u]_{B^\beta_{2,\infty}}$, which is a Banach space. Define $B^\beta_{2,2} := \{u \in L^2(K^{\boldsymbol{l}}, \mu) : \|u\|_{B^\beta_{2,2}} < \infty\}$ with the norm $\|u\|_{B^\beta_{2,2}} := \|u\|_{L^2(K^{\boldsymbol{l}},\mu)} + [u]_{B^\beta_{2,2}}$, which is also a Banach space. Now $B^\beta_{2,\infty}$, $B^\beta_{2,2}$ can be regarded as general Besov spaces with the scaling functions φ and ψ.

Using Kigami's method in Ref. 20, one can construct a local regular Dirichlet form on $K^{\boldsymbol{l}}$ as in Sec. 5 of Ref. 18. That is, there exists a limiting form

$$\begin{cases} \mathcal{E}_{\text{loc}}(u, u) = \mathcal{E}^{\beta^*}_{2,\infty}(u), \\ \mathcal{F} = \{u \in L^2(K^{\boldsymbol{l}}, \mu) : \mathcal{E}^{\beta^*}_{2,\infty}(u) < \infty\}, \end{cases} \tag{13}$$

where

$$\mathcal{E}^\beta_{2,\infty}(u) := \sup_{n\ge 0} \varphi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \times \sum_{x,y \in V^{\boldsymbol{l}}_w; |w|=n} |u(x) - u(y)|^2, \tag{14}$$

which will be discussed in Sec. 5. Define the semi-norm $\mathcal{E}^\beta_{2,2}$ by

$$\mathcal{E}^\beta_{2,2}(u) := \sum_{n=0}^\infty \varphi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \times \sum_{x,y \in V^{\boldsymbol{l}}_w; |w|=n} |u(x) - u(y)|^2. \tag{15}$$

We will see that $\mathcal{E}^\beta_{2,\infty}, \mathcal{E}^\beta_{2,2}$ are exactly discrete characterizations of spaces $B^\beta_{2,\infty}, B^\beta_{2,2}$ respectively.

The following two theorems are the main contribution of this paper.

Theorem 1.1. *Let $K^{\boldsymbol{l}}$ be the level-$\boldsymbol{l}$ scale irregular Sierpiński gasket for some infinite sequence $\boldsymbol{l}$ and $u \in L^2(K^{\boldsymbol{l}}, \mu)$. Assume that β satisfies* condition (B) (*see Definition* 2.3), *then we have*

$$\mathcal{E}^\beta_{2,\infty}(u) \asymp [u]^2_{B^\beta_{2,\infty}}, \tag{16}$$

$$\mathcal{E}^\beta_{2,2}(u) \asymp [u]^2_{B^\beta_{2,2}}. \tag{17}$$

The formulas (16) and (17) were proved on the Sierpiński gasket (see Theorem 1.1 of Ref. 21 and Theorem 1.1 of Ref. 3, respectively).

Theorem 1.2. *Let $K^{\boldsymbol{l}}$ be the level-$\boldsymbol{l}$ scale irregular Sierpiński gasket for some infinite sequence $\boldsymbol{l}$ and $u \in L^2(K^{\boldsymbol{l}}, \mu)$. Then there exists a positive constant C such that, for all $u \in \mathcal{F}$,*

$$\begin{aligned} C^{-1}\mathcal{E}^{\beta^*}_{2,\infty}(u) &\le \liminf_{\beta\uparrow\beta^*} (\beta^* - \beta)\mathcal{E}^\beta_{2,2}(u) \\ &\le \limsup_{\beta\uparrow\beta^*} (\beta^* - \beta)\mathcal{E}^\beta_{2,2}(u) \\ &\le C\mathcal{E}^{\beta^*}_{2,\infty}(u). \end{aligned} \tag{18}$$

Let us sketch the idea of the proof of these two theorems. First, using the Morrey–Sobolev inequality (Lemma 2.5), we obtain the equivalence result in Theorem 1.1. Second, we prove Theorem 1.2 using a monotonicity property. So, we immediately obtain the convergence of the $B^\beta_{2,2}$-norms to the $B^{\beta^*}_{2,\infty}$-norm as $\beta \uparrow \beta^*$ in a scale irregular Sierpiński gasket by applying Theorem 1.1 to Theorem 1.2 (as condition (B) is satisfied when β is close to β^*).

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We remark that the main difference between our result and previous works[1,3] is that, in our setting μ is not necessarily Ahlfors regular, thus $K^{\boldsymbol{l}}$ is not necessarily self-similar (by Theorem 2.1 of Ref. 22, the Hausdorff measure of a self-similar set is always Ahlfors regular, should it exist). This influences our form of semi-norms in the convergence theorem, which is different from the usual case in Theorem 1.3 of Ref. 3.

This paper is arranged as follows. In Sec. 2, we obtain a scaling property of ψ and φ (Lemma 2.1) and establish the Morrey–Sobolev inequality for those exponents β satisfying condition (B) (Lemma 2.5). By virtue of Lemma 2.5, we prove (16) of Theorem 1.1 in Sec. 3 and (17) of Theorem 1.1 in Sec. 4. In Sec. 5, we prove Theorem 1.2, which exactly means that a sequence of non-local Dirichlet forms converges to a local Dirichlet form.

Notation. The letters C, C_i, c, c_i are universal positive constants depending only on $K^{\boldsymbol{l}}$ which may vary at each occurrence. The sign $\asymp$ means that both $\leq$ and $\geq$ are true with uniform values of C depending only on $K^{\boldsymbol{l}}$. $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$.

2. THE MORREY–SOBOLEV INEQUALITY

We start by showing a scaling property of function $\psi(r), \varphi(r)$ with $r > 0$. A similar scaling condition was introduced in Sec. 2 of Ref. 23 on metric measure spaces, which was also used in Assumption 2.2 of Ref. 17 on homogeneous random fractals.

Lemma 2.1. *Let $K^{\boldsymbol{l}}$ be the level-$\boldsymbol{l}$ scale irregular Sierpiński gasket for some infinite sequence $\boldsymbol{l}$, we have that the following scaling condition holds for $\psi(r)$, for all $0 < r < R \leq 1$:*

$$c\left(\frac{R}{r}\right)^{\alpha_a \wedge \alpha_b} \leq \frac{\psi(R)}{\psi(r)} \leq C\left(\frac{R}{r}\right)^{\alpha_a \vee \alpha_b} \tag{19}$$

for some positive constants $0 < c \leq 1 \leq C$, where α_a, α_b are given by (2).

Moreover, we also have the following scaling condition holds for $\varphi(r)$, for all $0 < r < R \leq 1$:

$$c\left(\frac{R}{r}\right)^{\beta_a \wedge \beta_b} \leq \frac{\varphi(R)}{\varphi(r)} \leq C\left(\frac{R}{r}\right)^{\beta_a \vee \beta_b} \tag{20}$$

for some positive constants $0 < c \leq 1 \leq C$, where β_a, β_b are given by (3).

Proof. For $0 \leq m < n$, picking $R \in (\rho_{m+1}, \rho_m]$ and $r \in (\rho_{n+1}, \rho_n]$, we have

$$\frac{\psi(R)}{\psi(r)} = \frac{m_a^{-[m]_a} m_b^{-[m]_b}}{m_a^{-[n]_a} m_b^{-[n]_b}} = \frac{a^{-\alpha_a [m]_a} b^{-\alpha_b [m]_b}}{a^{-\alpha_a [n]_a} b^{-\alpha_b [n]_b}}.$$

Since $a, b > 1$, we have

$$\frac{\psi(R)}{\psi(r)} \leq \left(\frac{a^{-[m]_a} b^{-[m]_b}}{a^{-[n]_a} b^{-[n]_b}}\right)^{\alpha_a \vee \alpha_b}$$

$$= \left(\frac{\rho_m}{\rho_n}\right)^{\alpha_a \vee \alpha_b} \leq C_1 \left(\frac{R}{r}\right)^{\alpha_a \vee \alpha_b}.$$

Similarly, we also have

$$\frac{\psi(R)}{\psi(r)} \geq \left(\frac{\rho_m}{\rho_n}\right)^{\alpha_a \wedge \alpha_b} \geq C_2 \left(\frac{R}{r}\right)^{\alpha_a \wedge \alpha_b}.$$

Next, we consider

$$\frac{\varphi(R)}{\varphi(r)} = \frac{t_a^{-[m]_a} t_b^{-[m]_b}}{t_a^{-[n]_a} t_b^{-[n]_b}} = \frac{a^{-\beta_a [m]_a} b^{-\beta_b [m]_b}}{a^{-\beta_a [n]_a} b^{-\beta_b [n]_b}}.$$

Since $a, b > 1$, we have

$$\frac{\varphi(R)}{\varphi(r)} \leq \left(\frac{a^{-[m]_a} b^{-[m]_b}}{a^{-[n]_a} b^{-[n]_b}}\right)^{\beta_a \vee \beta_b}$$

$$= \left(\frac{\rho_m}{\rho_n}\right)^{\beta_a \vee \beta_b} \leq C_3 \left(\frac{R}{r}\right)^{\beta_a \vee \beta_b}.$$

Similarly, we also have

$$\frac{\varphi(R)}{\varphi(r)} \geq \left(\frac{\rho_m}{\rho_n}\right)^{\beta_a \wedge \beta_b} \geq C_4 \left(\frac{R}{r}\right)^{\beta_a \wedge \beta_b}.$$

The proof is complete. □

As any metric ball $B(x, r)$ in $K^{\boldsymbol{l}}$ contains a level-$(m + 2)$ cell and is contained in a level-$((m - 2) \vee 0)$ cell, by Lemma 2.1, we immediately derive the following.

Corollary 2.2. *For any $0 < r \leq 1$,*

$$c_1 \psi(r) \leq \mu(B(x, r)) \leq c_2 \psi(r), \tag{21}$$

where $c_1, c_2 > 0$ are independent of x and r.

The domain of Dirichlet forms on $K^{\boldsymbol{l}}$ can always be regarded as a subspace of $C(K^{\boldsymbol{l}})$, which is fundamental for discretization, by the following analogue of the Morrey–Sobolev inequality as in Refs. 3 and 5. The proof is adapted from Theorem 4.11(iii) of Ref. 24, and we will need another type condition on the exponent β as follows.

Definition 2.3. We say that β satisfies *condition (B)* if $\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b) > 0$ and $\beta > \alpha$.

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Remark 2.4. Indeed, the walk dimension is not less than 2 on nested fractals (see Remark below Theorem 6.1 in Ref. 25). Therefore, for our use, this *condition (B)* is satisfied for those β close enough to β^* since $\beta_l \geq 2 > \alpha_l$ for any $l \in \mathbb{N}\backslash\{1\}$. For example, it is easy to verify that *condition (B)* is satisfied for those β close enough to β^* on SG($\boldsymbol{l}$) which is generated by SG(2) and SG(3), since $\beta_2 = \frac{\log 5}{\log 2}, \beta_3 = \frac{\log 90/7}{\log 3}$.

Lemma 2.5 (Morrey–Sobolev inequality). *Let $K^{\boldsymbol{l}}$ be a level-$\boldsymbol{l}$ scale irregular Sierpiński gasket. Assume that β satisfies* condition (B), *then there exists a continuous version $\tilde{u} \in C(K^{\boldsymbol{l}})$ satisfying $\tilde{u} = u$ μ-almost everywhere in $K^{\boldsymbol{l}}$ and*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C\varphi(|x-y|)^{\frac{\beta}{2\beta^*}}\psi(|x-y|)^{-1/2}[u]_{B^{\beta}_{2,\infty}}, \quad (22)$$

for all $x, y \in K^{\boldsymbol{l}}$ with $|x-y| \leq 1/3$, where C is a positive constant independent of u.

Proof. For any $u \in B^{\beta}_{2,\infty}$ and $x, y \in K^{\boldsymbol{l}}$, let $r = |x-y| \leq 1/3$. Letting $B_1 = B(x,r)$, $B_2 = B(y,r)$ (recall that this stands for the metric ball), we define

$$u_r(x) = \frac{1}{\mu(B_1)}\int_{B_1} u(\xi)d\mu(\xi) = \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2} u(\xi)d\mu(\eta)d\mu(\xi),$$

and

$$u_r(y) = \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2} u(\eta)d\mu(\eta)d\mu(\xi).$$

First, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}|u_r(x) - u_r(y)|^2 &= \left\{\frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2}(u(\xi) - u(\eta))d\mu(\eta)d\mu(\xi)\right\}^2 \\ &\leq \frac{1}{\mu(B_1)\mu(B_2)}\int_{B_1}\int_{B_2}|u(\xi) - u(\eta)|^2 d\mu(\eta)d\mu(\xi) \\ &\leq C_1\psi(r)^{-2}\int_{K^{\boldsymbol{l}}}\left[\int_{B(\xi,3r)}|u(\xi) - u(\eta)|^2 d\mu(\eta)\right]d\mu(\xi) \\ &\leq C_2\psi(r)^{-1}\varphi(r)^{\frac{\beta}{\beta^*}}[u]^2_{B^{\beta}_{2,\infty}}, \end{aligned} \quad (23)$$

where we have used (7) in the third line and (11), Lemma 2.1 in the fourth line.

Next, let L be the set of Lebesgue points of u, and pick $x \in L$. Define $r_0 = r$, $r_k = a^{-[k]_a}b^{-[k]_b}r$, we have $r_k + r_{k+1} \leq r_k + (a \wedge b)^{-1}r_k < 2r_k$ for $k \geq 0$. By (23), we can similarly show that, for any $u \in B^{\beta}_{2,\infty}$,

$$|u_{r_k}(x) - u_{r_{k+1}}(x)|^2 \leq C_3\psi(r_k)^{-1}\varphi(r_k)^{\frac{\beta}{\beta^*}}[u]^2_{B^{\beta}_{2,\infty}}. \quad (24)$$

It follows that

$$\begin{aligned}|u(x) - u_r(x)| &\leq \sum_{k=0}^{\infty}|u_{r_k}(x) - u_{r_{k+1}}(x)| \\ &\leq C_3\sum_{k=0}^{\infty}\varphi(r_k)^{\frac{\beta}{2\beta^*}}\psi(r_k)^{-1/2}[u]_{B^{\beta}_{2,\infty}} \\ &\leq C_4\sum_{k=0}^{\infty}(a^{-[k]_a}b^{-[k]_b})^{\frac{\beta}{2\beta^*}(\beta_a\wedge\beta_b)}\varphi(r)^{\frac{\beta}{2\beta^*}} \\ &\quad\times(a^{-[k]_a}b^{-[k]_b})^{-\frac{1}{2}(\alpha_a\vee\alpha_b)}\psi(r)^{-1/2}[u]_{B^{\beta}_{2,\infty}} \\ &\leq C_5\varphi(r)^{\frac{\beta}{2\beta^*}}\psi(r)^{-1/2}[u]_{B^{\beta}_{2,\infty}}, \end{aligned} \quad (25)$$

where $C_5 = C_4\sum_{k=0}^{\infty}(a \wedge b)^{-\frac{k}{2}(\frac{\beta}{\beta^*}(\beta_a\wedge\beta_b)-(\alpha_a\vee\alpha_b))}$ is finite by condition (B) and we have used Lemma 2.1 in the second line. Combining (23) and (25), we derive

$$\begin{aligned}|u(x) - u(y)| &\leq |u(x) - u_r(x)| + |u_r(x) - u_r(y)| + |u_r(y) - u(y)| \\ &\leq C\varphi(r)^{\frac{\beta}{2\beta^*}}\psi(r)^{-1/2}[u]_{B^{\beta}_{2,\infty}}, \end{aligned} \quad (26)$$

for all x, $y \in L$.

Finally, for any $u \in B^{\beta}_{2,\infty}$, we use a standard procedure to obtain a continuous version $\tilde{u} \in C(K^{\boldsymbol{l}})$. For $x \in L$, let $\tilde{u}(x) = u(x)$. Since u is measurable in $K^{\boldsymbol{l}}$, by Theorem 1.8 of Ref. 26, $\mu(K^{\boldsymbol{l}}\backslash L) = 0$, which implies that L is dense in $K^{\boldsymbol{l}}$ (otherwise there exists a metric ball in $K^{\boldsymbol{l}}\backslash\overline{L}$ with positive μ-measure). Therefore, for any $x \in K^{\boldsymbol{l}}\backslash L$, one can find

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a sequence $\{x_n\} \subseteq L$ that converges to x. By (26), any such sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so we can define $\tilde{u}(x) = \lim_{n\to\infty} u(x_n)$ which does not depend on the choice of $\{x_n\}_{n\geq 1}$. Now $\tilde{u} = u$ μ-almost everywhere in $K^{\boldsymbol{l}}$ and it follows that $[u]_{B^\beta_{2,\infty}} = [\tilde{u}]_{B^\beta_{2,\infty}}$. Now for any $x, y \in K^{\boldsymbol{l}}$, pick $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq L$ that converge to x, y respectively. Then as

$$\frac{|\tilde{u}(x_n) - \tilde{u}(y_n)|}{\varphi(|x_n - y_n|)^{\frac{\beta}{2\beta^*}}\psi(|x_n - y_n|)^{-1/2}} = \frac{|u(x_n) - u(y_n)|}{\varphi(|x_n - y_n|)^{\frac{\beta}{2\beta^*}}\psi(|x_n - y_n|)^{-1/2}} \leq C[u]_{B^\beta_{2,\infty}} = C[\tilde{u}]_{B^\beta_{2,\infty}},$$

we obtain the desired inequality (22) and $\tilde{u} \in C(K^{\boldsymbol{l}})$ by letting $n \to \infty$. □

From now on, we will regard $B^\beta_{2,\infty} = B^\beta_{2,\infty} \cap C(K^{\boldsymbol{l}})$ and $B^\beta_{2,2} = B^\beta_{2,2} \cap C(K^{\boldsymbol{l}})$ by Lemma 2.5, that is, we admit that all the functions in $B^\beta_{2,\infty}, B^\beta_{2,2}$ are continuous.

3. DISCRETIZING $B^{\boldsymbol{\beta}}_{\boldsymbol{2,\infty}}$

In this section, we will prove (16) by deriving upper bound estimate of $[u]_{B^\beta_{2,\infty}}$ in Lemma 3.1 and lower bound estimate of $[u]_{B^\beta_{2,\infty}}$ in Lemma 3.2. For simplicity, denote

$$E_n(u) := \sum_{x,y\in V^{\boldsymbol{l}}_w, |w|=n} |u(x) - u(y)|^2. \tag{27}$$

Lemma 3.1. *Assume that β satisfies* condition (B), *then there exists a positive constant C such that, for all $u \in B^\beta_{2,\infty}$,*

$$[u]^2_{B^\beta_{2,\infty}} \leq C\mathcal{E}^\beta_{2,\infty}(u).$$

Proof. We will use the fact that

$$[u]^2_{B^\beta_{2,\infty}} = \sup_{r>0} h(r) = \sup_{0<r\leq 1} h(r),$$

where

$$h(r) := \varphi(r)^{-\frac{\beta}{\beta^*}}\psi(r)^{-1} \times \int_{K^{\boldsymbol{l}}}\int_{B(x,r)} |u(x) - u(y)|^2 d\mu(y)d\mu(x).$$

This simply follows from the fact that $h(r) = h(1)$ when $r \geq 1$, as $\varphi(r) = \varphi(1), \psi(r) = \psi(1)$ when $r \geq 1$ by definition, and the diameter of $K^{\boldsymbol{l}}$ is always assumed to be 1.

For two integers $m > n \geq 0$, denote by μ_m the Borel measure on $V^{\boldsymbol{l}}_m$ and let $\mu_m = \frac{1}{|V^{\boldsymbol{l}}_m|}\sum_{a\in V^{\boldsymbol{l}}_m} \delta_a$, where δ_a is the Dirac measure at point a and $|A|$ denotes the cardinality of a finite set A. Noting that μ_m weak $*$-converges to μ, we will first consider

$$I_{m,n}(u) := \int_{K^{\boldsymbol{l}}}\int_{B(x,r)} |u(x) - u(y)|^2 d\mu_m(y)d\mu_m(x) \leq \int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)} |u(x) - u(y)|^2 d\mu_m(y)d\mu_m(x) \tag{28}$$

for $\rho_{n+1} < r \leq \rho_n$. For any $x, y \in K^{\boldsymbol{l}}$ with $|x - y| \leq \rho_n$ where $x \in K^{\boldsymbol{l}}_w$ for some $w \in W^{\boldsymbol{l}}_n$, since $K^{\boldsymbol{l}}$ is connected, there exist words $\tilde{w} \in W^{\boldsymbol{l}}_n$ such that $y \in K^{\boldsymbol{l}}_{\tilde{w}}$ and $K^{\boldsymbol{l}}_{\tilde{w}} \cap K^{\boldsymbol{l}}_w \neq \emptyset$. Then we obtain

$$\begin{aligned} I_{m,n}(u) &\leq \sum_{|w|=n,|\tilde{w}|=n} \int_{K^{\boldsymbol{l}}_w}\int_{K^{\boldsymbol{l}}_{\tilde{w}}} |u(x) - u(y)|^2 \times d\mu_m(y)d\mu_m(x) \\ &= \sum_{|w|=n,|\tilde{w}|=n}\ \sum_{x\in K^{\boldsymbol{l}}_w\cap V^{\boldsymbol{l}}_m}\ \sum_{y\in K^{\boldsymbol{l}}_{\tilde{w}}\cap V^{\boldsymbol{l}}_m} \frac{1}{|V^{\boldsymbol{l}}_m|^2} \times |u(x) - u(y)|^2. \end{aligned}$$

For every pair $(w, \widetilde{w}) \in W^{\boldsymbol{l}}_n \times W^{\boldsymbol{l}}_n$, we can find a common vertex z of $V^{\boldsymbol{l}}_{\tilde{w}}$ and $V^{\boldsymbol{l}}_w$. Moreover, using $|u(x) - u(y)|^2 \leq 2(|u(x) - u(z)|^2 + |u(z) - u(y)|^2)$, we have

$$\begin{aligned} I_{m,n}(u) &\leq 2\sum_{|w|=n,|\tilde{w}|=n}\ \sum_{x\in K^{\boldsymbol{l}}_w\cap V^{\boldsymbol{l}}_m}\ \sum_{y\in K^{\boldsymbol{l}}_{\tilde{w}}\cap V^{\boldsymbol{l}}_m} \frac{1}{|V^{\boldsymbol{l}}_m|^2} \times (|u(x) - u(z)|^2 + |u(z) - u(y)|^2) \\ &\leq C_1 \sum_{|w|=n}\ \sum_{x\in K^{\boldsymbol{l}}_w\cap V^{\boldsymbol{l}}_m}\ \sum_{z\in V^{\boldsymbol{l}}_w} \frac{|K^{\boldsymbol{l}}_{\tilde{w}} \cap V^{\boldsymbol{l}}_m|}{|V^{\boldsymbol{l}}_m|^2} \times |u(x) - u(z)|^2 \\ &\leq C_1\psi(\rho_n)\psi(\rho_m) \times \sum_{|w|=n}\ \sum_{x\in K^{\boldsymbol{l}}_w\cap V^{\boldsymbol{l}}_m}\ \sum_{z\in V^{\boldsymbol{l}}_w} |u(x) - u(z)|^2, \end{aligned}$$

where we have used $|V^{\boldsymbol{l}}_m| \asymp \psi(\rho_m)^{-1}$ and $|K^{\boldsymbol{l}}_{\tilde{w}} \cap V^{\boldsymbol{l}}_m| \asymp \psi(\rho_m)^{-1}\psi(\rho_n)$ in the third line.

Then we estimate $|u(x) - u(z)|^2$. For every $x \in K^{\boldsymbol{l}}_w \cap V^{\boldsymbol{l}}_m$, $z \in V^{\boldsymbol{l}}_w$ where $w \in W^{\boldsymbol{l}}_n$, we pick (and fix) a decreasing sequence of cells $\{K^{\boldsymbol{l}}_{w_k}\}^m_{k=n}$ such that $|w_k| = k$ with $z \in K^{\boldsymbol{l}}_{w_n} \cap V^{\boldsymbol{l}}_n$, $x \in K^{\boldsymbol{l}}_{w_m} \cap$

$V_m^{\boldsymbol{l}}$. Then we obtain a sequence of vertices $\{z = x_n, x_{n+1}, \ldots, x_m = x\}$ where $x_k \in K_{w_k}^{\boldsymbol{l}} \cap V_k^{\boldsymbol{l}}$ for $k = n, \ldots, m$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|u(z) - u(x)|^2 &\leq \left(\sum_{k=n}^{m-1} \psi(\rho_n)^{-1}\psi(\rho_k)\right)\\ &\quad\times\left(\sum_{k=n}^{m-1} \psi(\rho_n)\psi(\rho_k)^{-1}|u(x_k) - u(x_{k+1})|^2\right)\\ &\leq C_2 \sum_{k=n}^{m-1} \psi(\rho_n)\psi(\rho_k)^{-1}|u(x_k) - u(x_{k+1})|^2.\end{aligned}$$

Noting that the cardinality of $(x, z) \in ((K_w^{\boldsymbol{l}} \cap V_m^{\boldsymbol{l}}) \times V_w^{\boldsymbol{l}})$ with $(p, q) := (x_k, x_{k+1})$ is not greater than $C'\psi(\rho_k)\psi(\rho_m)^{-1}$ for some $C' > 0$, we see that

$$\begin{aligned}I_{m,n}(u) &\leq C_3\psi(\rho_n)\psi(\rho_m)\\ &\quad\times\sum_{|w|=n}\ \sum_{(x,z)\in(K_w^{\boldsymbol{l}}\cap V_m^{\boldsymbol{l}})\times V_w^{\boldsymbol{l}}}\ \sum_{k=n}^{m-1}\psi(\rho_n)\psi(\rho_k)^{-1}\\ &\quad\times|u(x_k) - u(x_{k+1})|^2\\ &= C_3\psi(\rho_n)\psi(\rho_m)\\ &\quad\times\sum_{|w|=n}\sum_{k=n}^{m-1}\sum_{\substack{|w'|=k\\ K_{w'}^{\boldsymbol{l}}\subset K_w^{\boldsymbol{l}}}}\ \sum_{\substack{(x,z)\in(K_{w'}^{\boldsymbol{l}}\cap V_m^{\boldsymbol{l}})\times V_w^{\boldsymbol{l}},\\ (x_k,x_{k+1})=(p,q)}}\psi(\rho_n)\\ &\quad\times\psi(\rho_k)^{-1}|u(p) - u(q)|^2\\ &\leq C_4\psi(\rho_n)\psi(\rho_m)\\ &\quad\times\sum_{|w|=n}\sum_{k=n}^{m-1}\sum_{\substack{|w'|=k\\ K_{w'}^{\boldsymbol{l}}\subset K_w^{\boldsymbol{l}}}}\ \sum_{p,q\in K_{w'}^{\boldsymbol{l}}\cap V_{k+1}^{\boldsymbol{l}}}\psi(\rho_m)^{-1}\\ &\quad\times\psi(\rho_n)|u(p) - u(q)|^2\\ &= C_4\psi(\rho_n)^2\\ &\quad\times\sum_{k=n}^{m-1}\sum_{|w|=n}\sum_{\substack{|w'|=k\\ K_{w'}^{\boldsymbol{l}}\subset K_w^{\boldsymbol{l}}}}\ \sum_{p,q\in K_{w'}^{\boldsymbol{l}}\cap V_{k+1}^{\boldsymbol{l}}}|u(p) - u(q)|^2\\ &\leq C_5\psi(\rho_n)^2\sum_{k=n}^{m-1}E_{k+1}(u)\\ &\leq C_5\psi(\rho_n)^2\sum_{k=n}^{m}E_k(u). \end{aligned} \tag{29}$$

Finally, it follows that

$$\begin{aligned}&\varphi(r)^{-\frac{\beta}{\beta^*}}\psi(r)^{-1}I_{m,n}(u)\\ &\quad= \varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}I_{m,n}(u)\\ &\quad\leq C_5\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{m}\varphi(\rho_k)^{\frac{\beta}{\beta^*}}\psi^{-1}(\rho_k)\\ &\qquad\times\left(\sup_{k\geq 0}\varphi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)E_k(u)\right)\\ &\quad\leq C_6\sum_{k=n}^{m}(a\wedge b)^{(n-k)(\frac{\beta}{\beta^*}(\beta_a\wedge\beta_b)-(\alpha_a\vee\alpha_b))}\\ &\qquad\times\sup_{k\geq 0}\varphi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)E_k(u)\\ &\quad\leq C_7\sup_{k\geq 0}\varphi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)E_k(u),\end{aligned}$$

where we have used condition (B) in the third line. By Lemma 2.5, the fact Lemma 3.2 of Ref. 5, and that $\mu_m \times \mu_m$ weak $*$-converges to $\mu \times \mu$ as $m \to \infty$, our conclusion follows from taking the supremum in n. □

Lemma 3.2. *Assume that* β *satisfies* condition (B), *then there exists a positive constant* C *such that for all* $u \in B_{2,\infty}^{\beta}$,

$$\mathcal{E}_{2,\infty}^{\beta}(u) \leq C[u]_{B_{2,\infty}^{\beta}}^2.$$

Proof. First, we consider $|u(p) - u(q)|^2$, for $p, q \in V_w^{\boldsymbol{l}}$. Integrating with respect to x and dividing by $\mu(K_w^{\boldsymbol{l}})$, we know that

$$\begin{aligned}E_n(u) &= \sum_{p,q\in V_w^{\boldsymbol{l}},|w|=n}|u(p) - u(q)|^2\\ &\leq 2\sum_{p,q\in V_w^{\boldsymbol{l}},p\neq q,|w|=n}\left(\frac{1}{\mu(K_w^{\boldsymbol{l}})}\int_{K_w^{\boldsymbol{l}}}|u(p) - u(x)|^2 + |u(x) - u(q)|^2 d\mu(x)\right)\\ &\leq 12\sum_{p\in V_w^{\boldsymbol{l}},|w|=n}\frac{1}{\mu(K_w^{\boldsymbol{l}})}\\ &\quad\times\int_{K_w^{\boldsymbol{l}}}|u(p) - u(x)|^2 d\mu(x).\end{aligned} \tag{30}$$

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For every $p \in V_w^{\boldsymbol{l}}$ with $|w| = n$, we can fix a decreasing sequence of cells $\{K_{w_k}^{\boldsymbol{l}}\}_{k=n}^m$ for

$$m > \left(\frac{\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b)\log(a \vee b)}{(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b))\log(a \wedge b)} + 1\right) n$$
$$\text{(where } n \text{ will be determined in (36)),} \tag{31}$$

such that $p \in \cap_{k=n}^m K_{w_k}^{\boldsymbol{l}}$ with $w_n = w$, $|w_k| = k$. Let $0 < \delta < \frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b)$, by the Cauchy–Schwarz inequality, we have for $x_k \in K_{w_k}^{\boldsymbol{l}}$,

$$\begin{aligned} &|u(p) - u(x_n)|^2 \\ &\quad \le 2|u(p) - u(x_m)|^2 + 2\left(\sum_{k=n}^{m-1} \rho_n^{-\delta}\rho_k^{\delta}\right) \\ &\quad\quad \times \left(\sum_{k=n}^{m-1} \rho_n^{\delta}\rho_k^{-\delta}|u(x_k) - u(x_{k+1})|^2\right) \\ &\quad \le 2|u(p) - u(x_m)|^2 \\ &\quad\quad + C_1\left(\sum_{k=n}^{m-1} \rho_n^{\delta}\rho_k^{-\delta}|u(x_k) - u(x_{k+1})|^2\right). \end{aligned} \tag{32}$$

Integrating (32) with respect to $x_k \in K_{w_k}^{\boldsymbol{l}}$ and dividing by $\mu(K_{w_k}^{\boldsymbol{l}})$, we obtain

$$\begin{aligned} &\frac{1}{\mu(K_{w_n}^{\boldsymbol{l}})}\int_{K_{w_n}^{\boldsymbol{l}}} |u(p) - u(x_n)|^2 d\mu(x_n) \\ &\quad \le \frac{2}{\mu(K_{w_m}^{\boldsymbol{l}})}\int_{K_{w_m}^{\boldsymbol{l}}} |u(p) - u(x_m)|^2 d\mu(x_m) \\ &\quad\quad + C_1\sum_{k=n}^{m-1}\frac{\rho_n^{\delta}\rho_k^{-\delta}}{\mu(K_{w_k}^{\boldsymbol{l}})\mu(K_{w_{k+1}}^{\boldsymbol{l}})} \\ &\quad\quad \times \int_{K_{w_k}^{\boldsymbol{l}}}\int_{K_{w_{k+1}}^{\boldsymbol{l}}} |u(x_k) - u(x_{k+1})|^2 \\ &\quad\quad \times d\mu(x_{k+1})d\mu(x_k). \end{aligned} \tag{33}$$

Applying Lemma 2.5 to $p,x_m \in K_{w_m}^{\boldsymbol{l}}$ and using the fact that $\mu(K_{w_k}^{\boldsymbol{l}}) \asymp \mu(K_{w_{k+1}}^{\boldsymbol{l}}) \asymp \psi(\rho_k)$ (by Lemma 2.1), it follows that

$$\begin{aligned} &\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{p \in V_w^{\boldsymbol{l}},|w|=n}\frac{2}{\mu(K_{w_m}^{\boldsymbol{l}})}\int_{K_{w_m}^{\boldsymbol{l}}}|u(p) \\ &\quad - u(x_m)|^2 d\mu(x_m) \\ &\quad \le C_2\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)[u]_{B_{2,\infty}^{\beta}}^2 \\ &\quad\quad \times \sum_{p \in V_w^{\boldsymbol{l}},|w|=n}\frac{1}{\mu(K_{w_m}^{\boldsymbol{l}})}\int_{K_{w_m}^{\boldsymbol{l}}}\varphi(|p - x_m|)^{\frac{\beta}{\beta^*}} \\ &\quad\quad \times \psi(|p - x_m|)^{-1}d\mu(x_m) \\ &\quad \le C_3\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\varphi(\rho_m)^{\frac{\beta}{\beta^*}}\psi(\rho_m)^{-1} \\ &\quad\quad \times |V_n^{\boldsymbol{l}}|[u]_{B_{2,\infty}^{\beta}}^2 \quad \text{(using condition (B))} \\ &\quad \le C_4\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\varphi(\rho_m)^{\frac{\beta}{\beta^*}}\psi(\rho_m)^{-1}[u]_{B_{2,\infty}^{\beta}}^2 \\ &\quad \le C_5(a \wedge b)^{-m(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b))} \\ &\quad\quad \times (a \vee b)^{n\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b)}[u]_{B_{2,\infty}^{\beta}}^2 \\ &\quad \le C_5(a \wedge b)^{-n(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b))} \\ &\quad\quad \times [u]_{B_{2,\infty}^{\beta}}^2 \quad \text{(using (31)).} \end{aligned} \tag{34}$$

On the other hand, for all $x_k \in K_{w_k}^{\boldsymbol{l}}, x_{k+1} \in K_{w_{k+1}}^{\boldsymbol{l}}$, we have $|x_k - x_{k+1}| \le \rho_k$, thus

$$\begin{aligned} &\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{p \in V_w^{\boldsymbol{l}},|w|=n}\sum_{k=n}^{m-1}\frac{\rho_n^{\delta}\rho_k^{-\delta}}{\mu(K_{w_k}^{\boldsymbol{l}})\mu(K_{w_{k+1}}^{\boldsymbol{l}})} \\ &\quad \times \int_{K_{w_k}^{\boldsymbol{l}}}\int_{K_{w_{k+1}}^{\boldsymbol{l}}}|u(x_k) - u(x_{k+1})|^2 \\ &\quad \times d\mu(x_{k+1})d\mu(x_k) \\ &\le \varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{m-1}\rho_n^{\delta}\rho_k^{-\delta}\psi(\rho_k)^{-2} \\ &\quad \times \sum_{p \in V_w^{\boldsymbol{l}},|w|=n}\int_{K_{w_k}^{\boldsymbol{l}}}\int_{K_{w_{k+1}}^{\boldsymbol{l}}}|u(x_k) - u(x_{k+1})|^2 \\ &\quad \times d\mu(x_{k+1})d\mu(x_k) \\ &\le C_6\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{m}\rho_n^{\delta}\rho_k^{-\delta}\psi(\rho_k)^{-2} \\ &\quad \times \int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_k)}|u(x) - u(y)|^2 d\mu(y)d\mu(x) \\ &\le C_7\rho_n^{\delta}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{\infty}\rho_k^{-\delta}\varphi(\rho_k)^{\frac{\beta}{\beta^*}} \\ &\quad \times \psi(\rho_k)^{-1}\cdot[u]_{B_{2,\infty}^{\beta}}^2 \\ &\le C_8\sum_{k=n}^{\infty}(a \wedge b)^{(n-k)(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b) - \delta)} \end{aligned}$$

$$\times [u]^2_{B^\beta_{2,\infty}} \le C_9[u]^2_{B^\beta_{2,\infty}} \quad \text{(by Lemma 2.1)}. \tag{35}$$

Now we estimate $\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)E_n(u)$. To do this, substituting (33) into (30), we have from (34) and (35) that

$$\begin{aligned}
&\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)E_n(u)\\
&\le 12\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\left(\sum_{p\in V^{\boldsymbol{l}}_w,|w|=n}\frac{2}{\mu(K^{\boldsymbol{l}}_{w_m})}\right.\\
&\quad\times\int_{K^{\boldsymbol{l}}_{w_m}}|u(p)-u(x_m)|^2d\mu(x_m)\\
&\quad+\sum_{p\in V^{\boldsymbol{l}}_w,|w|=n}\sum_{k=n}^{m-1}\frac{\rho_n^\delta\rho_k^{-\delta}}{\mu(K^{\boldsymbol{l}}_{w_k})\mu(K^{\boldsymbol{l}}_{w_{k+1}})}\\
&\quad\times\int_{K^{\boldsymbol{l}}_{w_k}}\int_{K^{\boldsymbol{l}}_{w_{k+1}}}|u(x_k)-u(x_{k+1})|^2\\
&\quad\left.\times d\mu(x_{k+1})d\mu(x_k)\right)\\
&\le C_{10}[u]^2_{B^\beta_{2,\infty}},
\end{aligned}\tag{36}$$

which completes the proof. □

4. DISCRETIZING $B^{\boldsymbol{\beta}}_{\mathbf{2,2}}$

In this section, we will give the proof of (17) by deriving upper bound estimate of $[u]_{B^\beta_{2,2}}$ in Lemma 4.2 and lower bound estimate of $[u]_{B^\beta_{2,2}}$ in Lemma 4.3. We will use the following assertion.

Proposition 4.1. *For all $u\in B^\beta_{2,2}$, we have*

$$\begin{aligned}
[u]^2_{B^\beta_{2,2}} &\asymp \sum_{n=0}^\infty \varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\\
&\quad\times\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2d\mu(y)d\mu(x).
\end{aligned}\tag{37}$$

Proof. Splitting the integral domain $r\in(0,1]$ into $r\in(\rho_{n+1},\rho_n](n\ge 0)$, we have

$$\begin{aligned}
[u]^2_{B^\beta_{2,2}} &= \int_0^1\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,r)}\varphi(r)^{-\frac{\beta}{\beta^*}}\psi(r)^{-1}\right.\\
&\quad\left.\times|u(x)-u(y)|^2d\mu(y)d\mu(x)\right)\frac{dr}{r}\\
&= \sum_{n=0}^\infty\int_{\rho_{n+1}}^{\rho_n}\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,r)}\varphi(r)^{-\frac{\beta}{\beta^*}}\psi(r)^{-1}\right.\\
&\quad\left.\times|u(x)-u(y)|^2d\mu(y)d\mu(x)\right)\frac{dr}{r}\\
&= \sum_{n=0}^\infty\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\\
&\quad\times\int_{\rho_{n+1}}^{\rho_n}\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,r)}|u(x)-u(y)|^2\right.\\
&\quad\left.\times d\mu(y)d\mu(x)\right)\frac{dr}{r}.
\end{aligned}\tag{38}$$

Therefore,

$$\begin{aligned}
[u]^2_{B^\beta_{2,2}} &\le \sum_{n=0}^\infty\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\\
&\quad\times\int_{\rho_{n+1}}^{\rho_n}\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2\right.\\
&\quad\left.\times d\mu(y)d\mu(x)\right)\frac{dr}{r}\\
&= \sum_{n=0}^\infty\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\log(\rho_n/\rho_{n+1})\\
&\quad\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2\right.\\
&\quad\left.\times d\mu(y)d\mu(x)\right)\\
&\le \log(a\vee b)\sum_{n=0}^\infty\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\\
&\quad\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2\right.\\
&\quad\left.\times d\mu(y)d\mu(x)\right).
\end{aligned}\tag{39}$$

On the other hand, we have

$$\begin{aligned}
[u]^2_{B^\beta_{2,2}} &\ge \sum_{n=0}^\infty\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\\
&\quad\times\int_{\rho_{n+1}}^{\rho_n}\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_{n+1})}|u(x)-u(y)|^2\right.
\end{aligned}$$

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$$\times d\mu(y)d\mu(x)\Bigg)\frac{dr}{r}$$

$$= \sum_{n=0}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}\log(\rho_n/\rho_{n+1})$$

$$\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_{n+1})}|u(x)-u(y)|^2\right.$$

$$\times d\mu(y)d\mu(x))$$

$$\geq \log(a\wedge b)\sum_{n=0}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}$$

$$\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_{n+1})}|u(x)-u(y)|^2\right.$$

$$\left.\times d\mu(y)d\mu(x)\right)$$

$$\geq \log(a\wedge b)\sum_{n=1}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}$$

$$\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2\right.$$

$$\left.\times d\mu(y)d\mu(x)\right). \tag{40}$$

Note that $K^{\boldsymbol{l}}$ satisfies the chain condition (see Definition 3.4 of Ref. 24), using the same argument as Corollary 2.2 of Ref. 3, we have that for all $n\geq 1$, there exists some positive constant $C(n)$ such that

$$\int_{K^{\boldsymbol{l}}}\int_{K^{\boldsymbol{l}}}|u(x)-u(y)|^2 d\mu(y)d\mu(x)$$

$$\leq C(n)\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}$$

$$\times\left(\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2 d\mu(y)d\mu(x)\right).$$

It follows by (40) that

$$[u]^2_{B^{\beta}_{2,2}} \geq C\sum_{n=0}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}$$

$$\times\int_{K^{\boldsymbol{l}}}\int_{B(x,\rho_n)}|u(x)-u(y)|^2 d\mu(y)d\mu(x), \tag{41}$$

for some constant $C>0$. Combining (39) and (41), we obtain (37). □

By (11) and (37), we immediately obtain

$$[u]^2_{B^{\beta}_{2,\infty}} \leq C[u]^2_{B^{\beta}_{2,2}}. \tag{42}$$

Recall the notation $E_k(\cdot), I_{m,n}(\cdot)$ in (27) and (28). We now present two estimates.

Lemma 4.2. *Assume that β satisfies* condition (B), *then there exists a positive constant C such that for all $u\in B^{\beta}_{2,2}$,*

$$[u]^2_{B^{\beta}_{2,2}} \leq C\mathcal{E}^{\beta}_{2,2}(u). \tag{43}$$

Proof. By (29), for $\rho_{n+1}<r\leq\rho_n$ and $m>n$, we have

$$\sum_{n=0}^{\infty}\varphi(r)^{-\frac{\beta}{\beta^*}}\psi(r)^{-1}I_{m,n}(u)$$

$$= \sum_{n=0}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)^{-1}I_{m,n}(u)$$

$$\leq C_1\sum_{n=0}^{\infty}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{k=n}^{m}E_k(u)$$

$$\leq C_1\sum_{k=0}^{\infty}\sum_{n=0}^{k}\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)E_k(u)$$

$$\leq C_2\sum_{k=0}^{\infty}\sum_{n=0}^{k}(a\wedge b)^{(n-k)(\frac{\beta}{\beta^*}(\beta_a\wedge\beta_b)-(\alpha_a\vee\alpha_b))}$$

$$\times\varphi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)E_k(u)$$

$$\leq C_3\sum_{k=0}^{\infty}\varphi(\rho_k)^{-\frac{\beta}{\beta^*}}\psi(\rho_k)E_k(u),$$

where in the fourth line we have used Lemma 2.1 and condition (B). Letting $m\to\infty$ and applying Fatou's lemma, we complete the proof. □

Lemma 4.3. *Assume that β satisfies* condition (B), *then there exists a positive constant C such that for all $u\in B^{\beta}_{2,2}$,*

$$\mathcal{E}^{\beta}_{2,2}(u) \leq C[u]^2_{B^{\beta}_{2,2}}. \tag{44}$$

Proof. We adopt the same estimation as the proof of Lemma 3.2 with some obvious adjustment. For each n, let m be an integer that satisfies (31). It follows from (34) that

$$\varphi(\rho_n)^{-\frac{\beta}{\beta^*}}\psi(\rho_n)\sum_{p\in V^{\boldsymbol{l}}_w,|w|=n}\frac{2}{\mu(K^{\boldsymbol{l}}_{w_m})}$$

$$\times \int_{K^{\boldsymbol{l}}_{w_m}} |u(p) - u(x_m)|^2 d\mu(x_m)$$
$$\leq C_1 (a \wedge b)^{-n(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b))} [u]^2_{B^{\beta}_{2,\infty}}.$$

Therefore,

$$\sum_{n=0}^{\infty} \varphi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n) \sum_{p \in V^{\boldsymbol{l}}_w, |w|=n} \frac{2}{\mu(K^{\boldsymbol{l}}_{w_m})}$$
$$\times \int_{K^{\boldsymbol{l}}_{w_m}} |u(p) - u(x_m)|^2 d\mu(x_m)$$
$$\leq C_1 \sum_{n=0}^{\infty} (a \wedge b)^{-n\left(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b)\right)} [u]^2_{B^{\beta}_{2,\infty}}$$
$$\leq C_2 [u]^2_{B^{\beta}_{2,2}} \text{ (using condition (B) and (42)).}$$

Similarly, by the third line of (35) and recall that $0 < \delta < \frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b)$, we have

$$\sum_{n=0}^{m} \varphi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n)$$
$$\times \sum_{p \in V^{\boldsymbol{l}}_w, |w|=n} \sum_{k=n}^{m-1} \frac{\rho_n^{\delta} \rho_k^{-\delta}}{\mu(K^{\boldsymbol{l}}_{w_k}) \mu(K^{\boldsymbol{l}}_{w_{k+1}})}$$
$$\times \int_{K^{\boldsymbol{l}}_{w_k}} \int_{K^{\boldsymbol{l}}_{w_{k+1}}} |u(x_k) - u(x_{k+1})|^2$$
$$\times d\mu(x_{k+1}) d\mu(x_k)$$
$$\leq C_3 \sum_{n=0}^{m} \varphi(\rho_n)^{-\frac{\beta}{\beta^*}} \psi(\rho_n)$$
$$\times \sum_{k=n}^{m} \rho_n^{\delta} \rho_k^{-\delta} \psi(\rho_k)^{-2}$$
$$\times \int_{K^{\boldsymbol{l}}} \int_{B(x,\rho_k)} |u(x) - u(y)|^2 d\mu(y) d\mu(x)$$
$$\leq C_3 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \varphi(\rho_k)^{-\frac{\beta}{\beta^*}} \varphi(\rho_k)^{\frac{\beta}{\beta^*}} \varphi(\rho_n)^{-\frac{\beta}{\beta^*}}$$
$$\times \psi(\rho_n) \rho_n^{\delta} \rho_k^{-\delta} \psi(\rho_k)^{-2}$$
$$\cdot \int_{K^{\boldsymbol{l}}} \int_{B(x,\rho_k)} |u(x) - u(y)|^2 d\mu(y) d\mu(x)$$
$$\leq C_4 \sum_{k=0}^{\infty} \varphi(\rho_k)^{-\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1}$$
$$\times \sum_{n=0}^{k} \left(\frac{\rho_n}{\rho_k}\right)^{-\left(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b) - \delta\right)}$$
$$\cdot \int_{K^{\boldsymbol{l}}} \int_{B(x,\rho_k)} |u(x) - u(y)|^2 d\mu(y) d\mu(x)$$
$$\leq C_5 \sum_{k=0}^{\infty} \varphi(\rho_k)^{-\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1}$$
$$\times \sum_{n=0}^{k} (a \wedge b)^{(n-k)\left(\frac{\beta}{\beta^*}(\beta_a \wedge \beta_b) - (\alpha_a \vee \alpha_b) - \delta\right)}$$
$$\cdot \int_{K^{\boldsymbol{l}}} \int_{B(x,\rho_k)} |u(x) - u(y)|^2 d\mu(y) d\mu(x)$$
$$\leq C_6 \sum_{k=0}^{\infty} \varphi(\rho_k)^{-\frac{\beta}{\beta^*}} \psi(\rho_k)^{-1}$$
$$\times \int_{K^{\boldsymbol{l}}} \int_{B(x,\rho_k)} |u(x) - u(y)|^2 d\mu(y) d\mu(x)$$
$$\leq C_7 [u]^2_{B^{\beta}_{2,2}}.$$

Then using Fatou's lemma, we prove the inequality (44). □

5. CONVERGENCE OF $B^{\beta}_{2,2}$ TO $B^{\beta}_{2,\infty}$

In this section, we prove Theorem 1.2. Let β be a real number that satisfies condition (B). For each positive integer n, we define a non-negative definite symmetric bilinear form a_n as

$$a_n(u, v) = \varphi(\rho_n)^{-1} \psi(\rho_n) \times \sum_{w \in W^{\boldsymbol{l}}_n} \sum_{p,q \in V^{\boldsymbol{l}}_w} (u(p) - u(q))(v(p) - v(q)),$$

where u, v are continuous real functions on $K^{\boldsymbol{l}}$. For simplicity, we write

$$a_n(u) := a_n(u, u).$$

Note that $\varphi(\rho_n)^{-1}\psi(\rho_n) = (\frac{t_a}{m_a})^{[n]_a} (\frac{t_b}{m_b})^{[n]_b}$. Using Lemma 4.2 and the Remark below it of Ref. 18, we see that a_n is coincident with the notation $\mathcal{E}_n$ in Sec. 5 of Ref. 18. Therefore, by Theorem 5.1 of Ref. 18, $a_n(u)$ is non-decreasing for positive integer n and

$$a_n(u) \uparrow a_{\infty}(u) := \lim_{n \to +\infty} \varphi(\rho_n)^{-1} \psi(\rho_n)$$

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$$\times \sum_{w\in W_n^{\boldsymbol{l}}} \sum_{p,q\in V_w^{\boldsymbol{l}}} (u(p)-u(q))^2$$
$$= \mathcal{E}_{2,\infty}^{\beta^*}(u). \quad (45)$$

Moreover, $a_\infty(u,v) := \lim_{n\to+\infty} \varphi(\rho_n)^{-1}\psi(\rho_n) \sum_{w\in W_n^{\boldsymbol{l}}} \sum_{p,q\in V_w^{\boldsymbol{l}}} (u(p)-u(q))(v(p)-v(q))$ (with domain $\{u : a_\infty(u,u) < \infty\}$) is a regular local Dirichlet form (see Theorem 5.2 of Ref. 18).

Proof of Theorem 1.2. By (15), we see that

$$(\beta^*-\beta)\mathcal{E}_{2,2}^{\beta}(u) = \beta^*\left(1-\frac{\beta}{\beta^*}\right)\sum_{n=0}^{\infty} \varphi(\rho_n)^{1-\frac{\beta}{\beta^*}} a_n(u).$$

For the right-hand side, using the elementary inequality $e^x \geq 1+x$ for $x \in \mathbb{R}$, we have from Lemma 2.1 that

$$\left(1-\frac{\beta}{\beta^*}\right)\sum_{n=0}^{\infty} \varphi(\rho_n)^{1-\frac{\beta}{\beta^*}} a_n(u)$$
$$\leq \left(1-\frac{\beta}{\beta^*}\right)\sum_{n=0}^{\infty} \varphi(\rho_n)^{1-\frac{\beta}{\beta^*}} a_\infty(u)$$
$$\leq C_1(1-(t_a\wedge t_b)^{\frac{\beta}{\beta^*}-1})$$
$$\times \sum_{n=0}^{\infty} (t_a\wedge t_b)^{n(\frac{\beta}{\beta^*}-1)} a_\infty(u)$$
$$= C_1 a_\infty(u).$$

Hence, combined with (45), we have

$$\limsup_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathcal{E}_{2,2}^{\beta}(u) \leq \beta^* C_1 \mathcal{E}_{2,\infty}^{\beta^*}(u). \quad (46)$$

For the other side, for all $A < a_\infty(u)$, there exists $N \geq 1$ such that for all $n > N$, we have $a_n(u) > A$. Then we similarly obtain

$$\left(1-\frac{\beta}{\beta^*}\right)\sum_{n=0}^{\infty} \varphi(\rho_n)^{1-\frac{\beta}{\beta^*}} a_n(u)$$
$$> \left(1-\frac{\beta}{\beta^*}\right)\sum_{n=N+1}^{\infty} \varphi(\rho_n)^{1-\frac{\beta}{\beta^*}} A$$
$$\geq C_2(1-(t_a\vee t_b)^{\frac{\beta}{\beta^*}-1})$$
$$\times \sum_{n=N+1}^{\infty} (t_a\vee t_b)^{n\left(\frac{\beta}{\beta^*}-1\right)} A$$
$$= C_2(t_a\vee t_b)^{(N+1)\left(\frac{\beta}{\beta^*}-1\right)} A \to C_2 A,$$

as $\beta \uparrow \beta^*$. It follows that

$$\liminf_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathcal{E}_{2,2}^{\beta}(u) \geq \beta^* C_2 A,$$

for all $A < a_\infty(u)$. Hence, we also obtain

$$\liminf_{\beta\uparrow\beta^*} (\beta^*-\beta)\mathcal{E}_{2,2}^{\beta}(u) \geq \beta^* C_2 \mathcal{E}_{2,\infty}^{\beta^*}(u). \quad (47)$$

By (46) and (47), the proof is complete. Note that the proof is still valid when $a_\infty(u) = \infty$. □

Note that $\mathcal{E}_{2,2}^{\beta}$ defined on the Sierpiński gasket SG(2) converges to $\mathcal{E}_{2,\infty}^{\beta^*}$ as $\beta \uparrow \beta^*$ in the sense of Mosco in Ref. 3. However, we do not use the similar method to obtain Mosco-convergence of Dirichlet form for general scale irregular Sierpiński gaskets, since the semi-norms $\mathcal{E}_{2,2}^{\beta}$ are not monotonic in β. In addition, we will further study the convergence of p-energy for the scale irregular Sierpiński gaskets in an upcoming paper based on Ref. 27.

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