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Modelling and estimation of system reliability under dynamic operating environments and lifetime ordering constraints

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ABSTRACT

Modern systems are operating under dynamic environments and the components therein often exhibit positively correlated lifetimes. Moreover, due to various practical reasons such as the load sharing mechanism, it is not uncommon that lifetimes of some components dominate the others within the same system. In this study, we first propose a statistical model for system reliability evaluation by jointly considering the correlated component lifetimes and the lifetime ordering constraints. In specific, the effects of the dynamic environments are incorporated by modelling the cumulative hazard function as an exponential dispersion process and the lifetime ordering constraints are modelled by truncating the support of the joint lifetime distribution. We then discuss the statistical inference based on the proposed model. The point estimates of the model parameters as well as the lifetime quantiles are obtained by the maximum likelihood method, and the confidence intervals are constructed by using the generalized pivots. Extensive simulation show that the proposed interval estimation procedures can achieve accurate coverage even under small sample sizes. Two real examples are used to illustrate the proposed modelling and estimation framework.

1. Introduction

1.1. Background and motivation

Modern complex systems normally consist of a variety of components. The traditional method to evaluate the system reliability is to use the reliability block diagram (RBD) [1,2], where the components are connected and drawn in parallel or series configuration. By assuming that the component lifetimes are independent, the system reliability can be readily obtained based on the components reliability. Although the RBD method is intuitive and simple to implement, its application in practice has to be very cautious.

One major concern of RBD comes from the independent assumption of the component lifetimes. This assumption is commonly imposed to simplify the reliability evaluation but it is untenable in many real applications. In practice, the ageing processes of components in the same system are often correlated and hence the components' lifetimes are statistically dependent. In fact, the dependent or correlated component failures have been observed in a variety of application areas including the software system [3], the distributed computing system [4], the medical system [5], and the network system [6]. The second assumption commonly adopted in the RBD procedure is that the components are operating under the time-invariant environment [e.g., 7–9]. Nevertheless, this assumption is neither appropriate in practice as the operating environmental conditions are inevitably changing over time. Typical examples include environmental variables such as temperature and humidity, which usually exhibit daily and seasonal patterns. As another example, the dissolved oxygen, the salinity and the PH level in the sea water are often different from day to night. If the systems are operating under such environments, the ageing processes of their components are dynamically changing as well. In summary, the estimation of system reliability can be significantly biased if the effects of component lifetime dependence and dynamic operating environment are neglected.

The research to be proposed is motivated by a real example from the automobile industry. The braking system is an important subsystem of a vehicle and it allows the drivers to slow down and stop the vehicle as they see fit. When the brake pedal is engaged, ultimately the brake pads function on the four wheels and halt the vehicle. Fig. 1 shows the lifetimes of the front and rear brake pads, which work in series, from a fleet of 35 taxis. These taxis were owned by the same company

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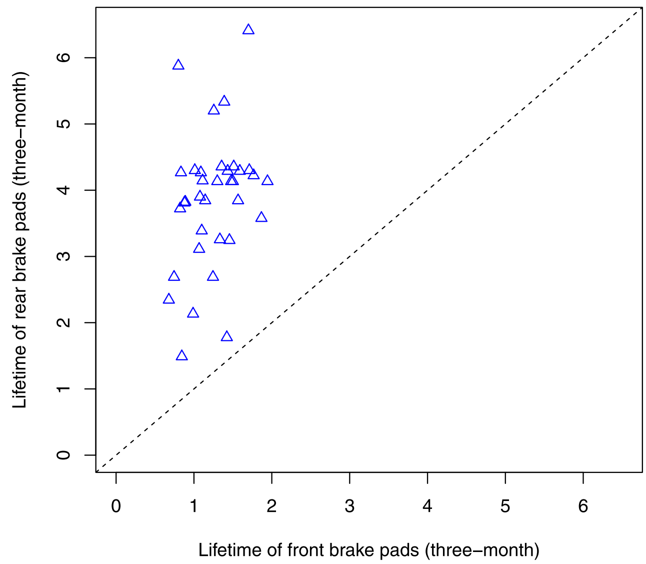

Fig. 1. Lifetime of rear brake pads as the function of the lifetime of the front brake pads.

and they were operated in the same city in 2013. Because the failure of the brake pads is mainly due to wear, the braking action plays the dominating role in affecting their lifetimes. In practice, the presence of braking depends on many factors such as road conditions, speed limits, obstructions and driving habit. In other words, the operating environment of the brake pads is time-varying. Moreover, the figure indicates a positive correlation between the lifetimes of the front and rear brake pads, as short/long lifetimes of the components often appear in pair. At last, the figure also suggests that the lifetime of the front pads is always shorter than the lifetime of the rear pads for each vehicle. This is reasonable as more vehicle weight is pushing downward on the front tyre in presence of braking, causing the wear of the front brake pads more severe. In fact, such a phenomena is known as ordering constraint in the literature and it is commonly seen in systems where one component has to share more load than the other.

1.2. Literature review

The component lifetime dependence, the dynamic operating environments and the lifetime ordering constraints are often considered separately in the literature. Generally speaking, there are two ways to deal with the correlations between the component lifetimes. The first method is to directly use the multivariate distributions such as the multivariate log-normal distribution [10], the multivariate Weibull distribution [11] and the multivariate Birnbaum–Saunders distribution [12]. Because the correlation structure is restrictive in these multivariate lifetime models, the copula functions have attracted more attention in the literature in constructing the multivariate distributions for their flexibility in modelling the correlations. Some recent examples include Liu et al. [13], Sun et al. [14], Lamboni and Kucherenko [15]. Nevertheless, one drawback of the copula-based methods is that the related physical interpretation is often lacking [16].

Secondly, the effects of dynamic operating environments were often incorporated by using the covariates information. For example, Okaro and Tao [17] used the Weibull proportional hazards model with the scale parameter being a function of the covariates. Izquierdo et al. [18] proposed the dynamic Weibull proportional hazards model and dynamic artificial neural network-based reliability model, where the dynamic information of the covariates was incorporated to the component failure rate. Li et al. [19] used the Bayesian networks and proportional hazards model for reliability assessment by treating the dynamic environments as covariates. In practice, however, the values of the covariates may not be available, e.g., the factors affecting the braking behaviour. In such cases, the most common way to model the effects of dynamic operating environments is to multiply the hazard function by a deterministic function of the calendar/operating time or a random variable following a certain distribution [20–22]. However, as stated in Hong et al. [23], such a treatment essentially overlooks the uncertainties in the dynamic environments.

At last, the lifetime ordering constraints have been studied by a multitude of studies. For example, the ordering of component lifetimes is often observed in the k-out-of-n systems or the coherent systems where some components have to share more load/weights than the others [24–26]. As shown in Zhang [26], ignorance of the lifetime ordering constraints could lead to a significant bias in the system reliability evaluation and hence could affect the decision making processes such as designing maintenance plan and warranty policies. The focus of these studies is mainly on estimation of the survival distributions, and neither of them considers the component lifetime dependence and the dynamic operating environments.

In summary, there have been very few studies simultaneously tackling the above-mentioned three factors. Two excellent exceptions are the recent studies by Hong et al. [23] and Xu et al. [16]. The underlying idea of these studies is to use the cumulative exposure model, where the equivalent operating time is modelled by a stochastic process to account for both the dynamic operating environments and the correlations between component lifetimes, and its outstanding performance in evaluating multi-component system reliabilities has been validated by simulations and real examples. Nevertheless, these methods cannot be easily extended to deal with the ordering constraints on the component lifetimes.

In terms of the system reliability evaluation, the point estimation is generally easy as the standard maximum likelihood procedures can be well employed. In practice, the interval estimation is of more interest as it can quantify uncertainties during point estimation. When the components are independently operating under the time-invariant environments, the confidence intervals of the system reliability are often obtained by analytically deriving the distribution for some constructed quantities [27–29]. However, an exact distribution is generally not easy to construct when the components lifetimes are correlated and the operating environments are dynamic. In such cases, the large-sample approximation [16] and the bootstrap-based methods [23] are often used. Nevertheless, their performance in presence of moderate sample sizes cannot be guaranteed.

1.3. Objectives and outline

By employing the brake pads lifetime data as an illustrative example, the objective of this study is two-fold. The first objective is to develop a general statistical model for system reliability evaluation by taking the dynamic operating environments, component lifetime dependence, and lifetime ordering constraints into consideration. Following the works in Hong et al. [23], Xu et al. [16] and assuming that the components are operated under the same dynamic operating environments, the first two factors are effectively incorporated by modelling the cumulative hazard function as a stochastic process. In particular, the exponential dispersion (ED) process which includes many commonly used stochastic processes as its special cases is adopted in this study. Furthermore, the support of the joint lifetime distribution is truncated to tackle the ordering constraints on the component lifetimes. Once the modelling framework is proposed, the second objective of this study is to develop the estimation procedures for the model parameters and the system reliability. In specific, the maximum likelihood method is used for the point estimation while the generalized inference methods are proposed for the interval estimation.

The rest of the paper is organized as follows. Section 2 introduces the modelling framework for components working under dynamic operating environments and with lifetime dependence and ordering constraints. Based on the proposed models, Section 3 develops the statistical inference procedures for estimating the system reliability. Extensive simulations are conducted in Section 4 to assess the performance of the proposed interval estimation methods. The analysis of the brake pads lifetime data is provided in Section 5, where another example on soccer data analysis is also discussed. Concluding remarks are given in Section 6.

2. Modelling of component lifetimes

Let T denote the component lifetime and $S(t)$ the survival function. Then $S(t)=\exp\{-C(t)\}$, where $C(t)$ is the cumulative hazard function. For many popular lifetime models such as the Weibull, log-normal and gamma distributions, $C(t)$ is simply a deterministic function of the calendar/operating time t. However, such a setting overlooks the unpredictable change in the operating environment. To account for the effects of the dynamic environment, it would be more reasonable to assume that $C(t)$ follows a stochastic process $\{\mathcal{Y}(t), t \geq 0\}$ [16,23]. In this paper, we assume that $\{\mathcal{Y}(t), t \geq 0\}$ is an ED process [30–32], denoted as $\mathcal{ED}(\mu H_\beta(t), \eta)$, which has the following properties: (i) $\mathcal{Y}(0)=0$; (ii) it has independent increments; (iii) each increment $Y=\mathcal{Y}(t)-\mathcal{Y}(s)$ $(s<t)$ follows the ED distribution $(ED(\mu\Delta H_\beta(t), \eta))$ with probability density function (PDF)

$$f_{ED}(y;\theta,\eta)=c(y,\Delta H_\beta(t),\eta)\exp\{\eta[y\theta-\Delta H_\beta(t)\kappa(\theta)]\}, \quad (1)$$

where $\Delta H_\beta(t)=H_\beta(t)-H_\beta(s)$, $H_\beta(\cdot)$ is a monotone increasing function of time with parameter β, $c(\cdot)$ is the canonical function, and $\kappa(\cdot)$ is the cumulant function related to the successive cumulants of the ED distribution. In addition, $\mu=\kappa'(\theta)$ is the drift parameter and η is the dispersion parameter. The mean and variance of ED process are respectively $\mathrm{E}[\mathcal{Y}(t)]=\mu H_\beta(t)$ and $\mathrm{Var}[\mathcal{Y}(t)]=\kappa''(\theta)H_\beta(t)/\eta$. When $\kappa''(\theta)=\mu^d$, $d\in(-\infty,0]\cup[1,\infty)$, ED process covers many popular stochastic processes including, for example, Wiener process ($d=0$), Poisson process ($d=1$), compound Poisson process ($1<d<2$), gamma process ($d=2$) and Inverse Gaussian process ($d=3$). $\kappa(\theta)$ has an analytical form as follows [33]:

$$\kappa(\theta)=\begin{cases}\exp(\theta), & d=1,\\ -\log(-\theta), & d=2,\\ \dfrac{[(1-d)\theta]^{(2-d)/(1-d)}}{2-d}, & d\neq 1 \text{ and } 2.\end{cases} \quad (2)$$

It can be easily shown that $\kappa(\theta)$ is a strictly convex function of θ given d due to $\kappa''(\theta)=\mu^d$ with $\mu>0$, and that the moment generating function of $\mathcal{Y}(t)$ is

$$\mathrm{E}[\exp(s\mathcal{Y}(t))]=\exp\{\eta H_\beta(t)[\kappa(\theta+s/\eta)-\kappa(\theta)]\}. \quad (3)$$

When there are two components as in the motivating example, the model can be written as

$$T_i|C(t)\sim S(t|C(t)),\ C(t)\sim\mathcal{ED}(\mu H_\beta(t),\eta)\quad i=1,2, \quad (4)$$

where T_i represents the lifetime for the ith component. It is worth mentioning that the original lifetime distribution of T_i can be any distribution with positive support, which generalizes the results of [16,23]. Based on model (4), the marginal survival function of T_i is $S_{T_i}(t)=\mathrm{E}[\exp(-C(t))]=\exp\{\eta H_\beta(t)[\kappa(\theta-1/\eta)-\kappa(\theta)]\}$. Thus, some special lifetime distributions are obtained for different forms of $H_\beta(t)$, for example, exponential distribution with $H_\beta(t)=t$, Weibull distribution with $H_\beta(t)=t^\beta$, Gompertz distribution with $H_\beta(t)=\exp(\beta t)-1$, and Lomax distribution with $H_\beta(t)=\log(1+t/\beta)$, etc. Under model (4), it can be shown that the joint survival function of T_1 and T_2 is

$$S(t_1,t_2)=\exp\left\{-\eta H_\beta(t_{(1)})\left[\kappa(\theta-1/\eta)-\kappa(\theta-2/\eta)\right]-\eta H_\beta(t_{(2)})\times\left[\kappa(\theta)-\kappa(\theta-1/\eta)\right]\right\},$$

where $t_{(1)}<t_{(2)}$ are the values by sorting t_1 and t_2 in an ascending order. In addition, the component lifetimes T_1 and T_2 are positive quadrant dependent. The derivations are given in Appendix.

Analogous to [16], we can show that $S(t_1,t_2)$ is not continuous, which will be singular when $t_1=t_2$, and that the joint PDF of T_1 and T_2 is

$$f(t_1,t_2)=\begin{cases}\eta^2\nabla\kappa_1\nabla\kappa_2\prod_{i=1}^2\dfrac{\partial H_\beta(t_i)}{\partial t_i}\exp\{-\eta H_\beta(t_1)\nabla\kappa_1-\eta H_\beta(t_2)\nabla\kappa_2\}, & t_1<t_2,\\ \eta^2\nabla\kappa_1\nabla\kappa_2\prod_{i=1}^2\dfrac{\partial H_\beta(t_i)}{\partial t_i}\exp\{-\eta H_\beta(t_2)\nabla\kappa_1-\eta H_\beta(t_1)\nabla\kappa_2\}, & t_1>t_2,\\ \eta(\nabla\kappa_2-\nabla\kappa_1)\dfrac{\partial H_\beta(t)}{\partial t}\exp\{-\eta H_\beta(t)\nabla\kappa_0\}, & t_1=t_2=t.\end{cases} \quad (5)$$

where $\nabla\kappa_0=\kappa(\theta)-\kappa(\theta-2/\eta)$, $\nabla\kappa_1=\kappa(\theta-1/\eta)-\kappa(\theta-2/\eta)$, and $\nabla\kappa_2=\kappa(\theta)-\kappa(\theta-1/\eta)$. The support of (T_1,T_2) is divided into three parts. The first part is the region with $T_1<T_2$, where the first component will have failed prior to the second component. While the second region with $T_1>T_2$ implies that the second component fails earlier than the first component. In the third region, the two components fail simultaneously due to the great influence from the external environments. Thus, model (4) fails to describe the phenomenon that T_1 is strictly smaller than T_2, as in the brake pads data case.

A direct solution is to truncate the support of $f(t_1,t_2)$, and to restrict it on the region of $t_1<t_2$. If T_1 is strictly smaller than T_2, then the joint PDF of T_1 and T_2 is

$$f_{TR}(t_1,t_2)=\eta^2\nabla\kappa_0\nabla\kappa_2\prod_{i=1}^2\frac{\partial H_\beta(t_i)}{\partial t_i}\times\exp\{-\eta H_\beta(t_1)\nabla\kappa_1-\eta H_\beta(t_2)\nabla\kappa_2\}, t_1<t_2. \quad (6)$$

The derivation is given in Appendix. When T_1 is strictly larger than T_2, the joint PDF of T_1 and T_2 could be derived similarly based on the case of $t_1>t_2$ in (5).

Although the new joint PDF can describe lifetime ordering constraints, we need to check whether $f_{TR}(t_1,t_2)$ keeps the correlation between T_1 and T_2, because the two components are affected by common stochastic environments, and neglecting dependence will underestimate or overestimate the system reliability [23,34]. One popular method is to investigate the order of total positivity between T_1 and T_2 [35]. Following the procedures in Appendix, we establish that T_1 and T_2 are totally positive of order 2, which indicates a strong dependence relationship and includes the positive quadrant dependence and the Pearson linear dependence as special cases [36, chap.9].

In summary, correlation and ordering between two random variables could be figured out simultaneously by $f_{TR}(t_1,t_2)$. Apparently, correlation between two random variables cannot be reflected by a single parameter in the proposed model. The classical Pearson's correlation coefficient is not sufficient to describe such relationship, because T_1 and T_2 are affected by a common ED process. In fact, we have shown that some nonlinear construction of correlation exists in the proposed model, and it may be difficult to quantify correlation by model parameters. In Section 5 of the real applications, the Spearman's rank correlation is used and tested for existence of dependence between two random variables.

3. Statistical inference of model parameters

Based on the model developed in Section 2, the next important task is to estimate the model parameters and construct their confidence intervals. The following two subsections discuss these two important tasks respectively.

3.1. Point estimation

Assume that the observed data are $\{(t_{1j}, t_{2j}),\ j = 1, \dots, n\}$, and $t_{1j} < t_{2j}$ for each j. Let $\boldsymbol{\theta} = (\eta, \theta, d, \beta)$. The likelihood function based on $f_{TR}(t_1, t_2)$ in (6) could be written as

$$L_1(\boldsymbol{\theta}) = \eta^{2n}[\nabla\kappa_0]^n[\nabla\kappa_2]^n \prod_{i=1}^{2}\prod_{j=1}^{n} H'_\beta(t_{ij}) \times \exp\left\{-\eta\nabla\kappa_1 \sum_{j=1}^{n} H_\beta(t_{1j}) - \eta\nabla\kappa_2 \sum_{j=1}^{n} H_\beta(t_{2j})\right\}, \tag{7}$$

where $H'_\beta(t_{ij}) = \frac{\partial H_\beta(t)}{\partial t}\Big|_{t=t_{ij}}$. Let $\lambda_1 = \eta\nabla\kappa_0$, $\lambda_2 = \eta\nabla\kappa_2$, and $\boldsymbol{\omega} = (\lambda_1, \lambda_2, \beta)$. Then the likelihood function (7) becomes

$$L_2(\boldsymbol{\omega}) = \lambda_2^n\lambda_1^n \prod_{i=1}^{2}\prod_{j=1}^{n} H'_\beta(t_{ij}) \times \exp\left\{-\lambda_1 \sum_{j=1}^{n} H_\beta(t_{1j}) - \lambda_2 \sum_{j=1}^{n}[H_\beta(t_{2j}) - H_\beta(t_{1j})]\right\}. \tag{8}$$

Maximizing $\log L_2(\boldsymbol{\omega})$ with respect to $\boldsymbol{\omega}$, we obtain the maximum likelihood estimators of λ_1 and λ_2 as follows:

$$\hat\lambda_1 = \frac{n}{\sum_{j=1}^{n} H_{\hat\beta}(t_{1j})},\quad \hat\lambda_2 = \frac{n}{\sum_{j=1}^{n}[H_{\hat\beta}(t_{2j}) - H_{\hat\beta}(t_{1j})]}.$$

The maximum likelihood estimator of β can be obtained by maximizing the following function:

$$-n\log\left(\sum_{j=1}^{n}[H_\beta(t_{2j}) - H_\beta(t_{1j})]\right) - n\log\left(\sum_{j=1}^{n} H_\beta(t_{1j})\right) + \sum_{i=1}^{2}\sum_{j=1}^{n}\log H'_\beta(t_{ij}).$$

Remark: Once the data are observed, the values of $(\hat\lambda_1, \hat\lambda_2)$ are fixed regardless of the forms of $\kappa(\cdot)$. Therefore, the original parameters η, θ and d cannot be simultaneously identified, and only two of them can be estimated based on the observed data. We recommend two ways to do statistical inference: choose a specified value of d or let $\eta = 1$. The former choice means that we choose a specified stochastic process $\{\mathcal{Y}(t), t \ge 0\}$ to incorporate the effects of dynamic environments into the model, while the latter case is to scale the dispersion parameter and let the data to learn an optimal d (corresponding to a stochastic process) to describe the stochastic effects of dynamic environment.

3.2. Interval estimation

As reviewed before, interval estimation based on the large-sample normal approximation or the bootstrap usually performs unsatisfactorily when the sample size is small. Here, we propose the generalized inference method to construct confidence intervals for the model parameters. To construct exact confidence interval (CI) of β, we need the following lemmas, which could be adapted from [37].

Lemma 1. *If $V_{1:n} < \cdots < V_{n:n}$ are order statistics from standard exponential distribution with sample size n, let $W_1 = nV_{1:n}$, $W_i = (n - i + 1)(V_{i:n} - V_{i-1:n})$, $i = 2, \dots, n$, then $W_1, \dots, W_n$ are independent standard exponential random variables.*

Lemma 2. *If $Q_i = \sum_{j=1}^{i} W_j$, $i = 1, \dots, n$, and $U_i = Q_i/Q_n, i = 1, \dots, n-1$, then $U_1 < \cdots < U_{n-1}$ are order statistics from uniform $(0, 1)$ distribution with sample size $n - 1$.*

Theorem 1. *Let $Z_1 = H_\beta(T_1)$ and $Z_2 = H_\beta(T_2) - H_\beta(T_1)$. Then $Z_1 \sim Exp(\lambda_1)$ and $Z_2 \sim Exp(\lambda_2)$, where $Exp(\lambda)$ denotes the exponential distribution with rate λ. Moreover, Z_1 and Z_2 are independent,*

See the proof in Appendix. From Theorem 1, we have that $\lambda_1 Z_1$ and $\lambda_2 Z_2$ follow standard exponential distribution and they are independent. Let $z_{1j} = H_\beta(t_{1j})$, $z_{2j} = H_\beta(t_{2j}) - H_\beta(t_{1j})$, $j = 1, \dots, n$. Denote that $z_{i:1:n} < \cdots < z_{i:n:n}$ are the ascending order of $z_{ij}, j = 1, \dots, n, i = 1, 2$. Let

$$W_{i1} = nz_{i:1:n},\ W_{ij} = (n - j + 1)(z_{i:j:n} - z_{i:j-1:n}), j = 2, \dots, n,\ i = 1, 2.$$
$$Q_{ij} = \sum_{k=1}^{j} W_{ik}, j = 1, \dots, n,\ U_{i:j} = Q_{ij}/Q_{in}, j = 1, \dots, n-1. \tag{9}$$

From Lemmas 1–2 and Theorem 1, we know that both $U_{1:1} < \cdots < U_{1:n-1}$ and $U_{2:1} < \cdots < U_{2:n-1}$ are the order statistics from the uniform $(0, 1)$ distribution with sample size $n-1$, and they are independent. Then we construct the pivotal quantity [38, chap.9] for the parameter β:

$$M(\beta) = \sum_{j=1}^{n-1}(-2\log U_{1:j}) + \sum_{j=1}^{n-1}(-2\log U_{2:j}) = M_1(\beta) + M_2(\beta). \tag{10}$$

It can be easily shown that $M_1(\beta)$ and $M_2(\beta)$ are strictly increasing function of β. Both $M_1(\beta)$ and $M_2(\beta)$ follow the χ^2 distribution with $2(n - 1)$ degrees of freedom, and they are independent. Thus, $M(\beta)$ has the χ^2 distribution with $4(n - 1)$ degrees of freedom. An exact $1 - \alpha$ CI for β is

$$\left[M^{-1}\left[\chi^2_{1-\alpha/2}(4(n-1))\right], M^{-1}\left[\chi^2_{\alpha/2}(4(n-1))\right]\right], \tag{11}$$

where $\chi^2_\alpha(m)$ is the upper α percentile of the χ^2 distribution with m degrees of freedom and $M^{-1}(x)$ is the solution in β of the equation $M(\beta) = x$, for $x > 0$.

For the parameter λ_1, we know that $E_1 = 2\lambda_1 \sum_{j=1}^{n} z_{1j} = 2\lambda_1 \sum_{j=1}^{n} H_\beta(t_{1j})$ has the χ^2 distribution with $2n$ degrees of freedom. Then,

$$\lambda_1 = \frac{E_1}{2\sum_{j=1}^{n} H_\beta(t_{1j})}. \tag{12}$$

Let $g(M, T)$ be the unique solution of $M(\beta) = x$. Then according to the substitution method [39], we substitute $g(M, T)$ for β in (12) and have the generalized pivotal quantity for the parameter λ_1:

$$P_1 = \frac{E_1}{2\sum_{j=1}^{n} H_{g(M,t)}(t_{1j})}, \tag{13}$$

where $t = \{(t_{1j}, t_{2j}), j = 1, \dots, n\}$ is the observed values of $T = \{(T_{1j}, T_{2j}), j = 1, \dots, n\}$. It is easy to see that the distribution of P_1 is free of any unknown parameters.

Notice that $E_2 = 2\lambda_2 \sum_{j=1}^{n} z_{2j} = 2\lambda_2 \sum_{j=1}^{n}[H_\beta(t_{2j}) - H_\beta(t_{1j})]$ has the χ^2 distribution with $2n$ degrees of freedom. Then we have that

$$\lambda_2 = \frac{E_2}{2\sum_{j=1}^{n}[H_\beta(t_{2j}) - H_\beta(t_{1j})]}. \tag{14}$$

Using the substitution method, we have the generalized pivotal quantity for the parameter λ_2:

$$P_2 = \frac{E_2}{2\sum_{j=1}^{n}[H_{g(M,t)}(t_{2j}) - H_{g(M,t)}(t_{1j})]}. \tag{15}$$

The distribution of P_2 is free of any unknown parameters.

For any function of the parameters β, λ_1 and λ_2, say, $J(\beta, \lambda_1, \lambda_2)$, the generalized pivotal quantity for $J(\beta, \lambda_1, \lambda_2)$ can be constructed as

$$P_J = J(g(M, t), P_1, P_2), \tag{16}$$

which is also free of any unknown parameters. Thus, an approximate $1 - \alpha$ CIs for the parameters λ_1, λ_2 and $J(\beta, \lambda_1, \lambda_2)$ can be obtained by the following Monte Carlo Algorithm 1.

4. Simulation studies

For illustration, simulation studies are performed to assess the proposed inference method. Assume that the system has two components, and the effects of random environments are described by ED process $\mathcal{ED}(\mu H_\beta(t), 1)$, where $H_\beta(t) = t^\beta$, and the parameter d for identifying model is set to be 1.5, 2, and 3 corresponding to compound Poisson

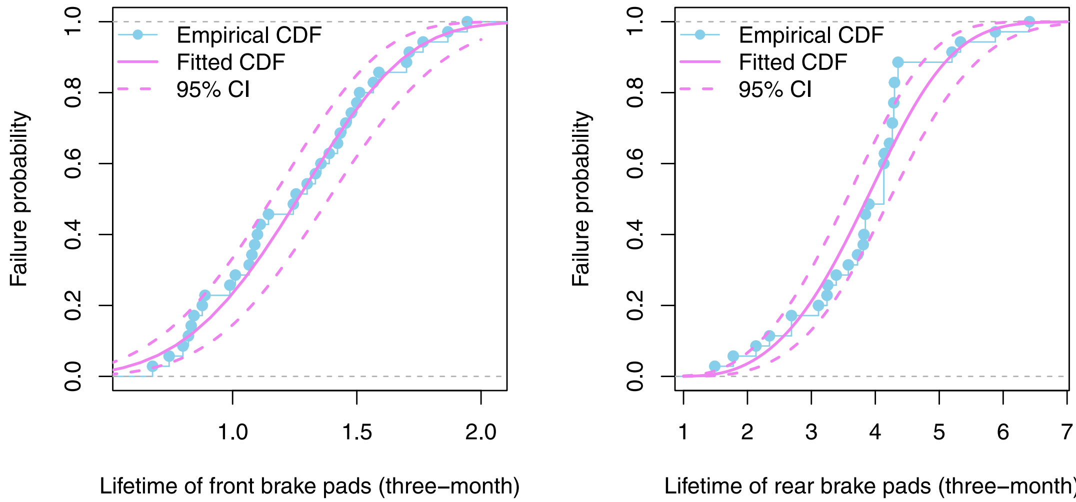

Fig. 2. Fitted marginal CDFs for the lifetime data of front and rear brake pads.

Algorithm 1: Generalized CIs for λ_1, λ_2 and $J(\beta, \lambda_1, \lambda_2)$

Input: The observed data $\{(t_{1j}, t_{2j}), j = 1, \cdots, n\}$.

1 Generate $M \sim \chi^2(4(n-1))$, $E_1 \sim \chi^2(2n)$ and $E_2 \sim \chi^2(2n)$ independently;

2 Compute the values of P_1, P_2 and P_J based on (13), (15) and (16), respectively;

3 Repeat the above two steps N times, and get N values of P_i: $P_{i1}, \cdots, P_{iN}$, $i = 1, 2, J$;

4 Let $P_{i,\alpha}$ denotes the upper α percentile of P_i. Then $P_{i,1-\alpha/2}$ and $P_{i,\alpha/2}$ can be obtained based on $P_{i1}, \cdots, P_{iN}$;

5 $[P_{i,1-\alpha/2}, P_{i,\alpha/2}]$ is an approximate $1-\alpha$ CI for the corresponding parameter.

process, gamma process and Inverse Gaussian process, respectively. Without loss of generality, let $\mu = 1$. Then the parameters $(\lambda_1, \lambda_2) =$ (1.000, 0.667), (1.098, 0.693) and (1.236, 0.732), respectively. β is chosen as 0.8 and 3 corresponding to decreasing and increasing failure rate, respectively. In addition, the sample size n is set as 20, 30 and 50.

For each combination of (n, d, β), we generate 2,000 samples to obtain the bias and mean square errors (MSE) of estimating the parameters as well the 0.05 and 0.95 lifetime quantiles of the two components (denoted as $t_{1,0.05}, t_{1,0.95}, t_{2,0.05}, t_{2,0.95}$). For better presentation, the results of relative bias (RB) and root MSE (RMSE), which are defined as

$$\text{RB} = \left| \frac{1}{2000} \sum_{i=1}^{2000} \frac{\hat{\theta}_i - \theta}{\theta} \right| \times 100, \quad \text{RMSE} = \sqrt{\frac{1}{2000} \sum_{i=1}^{2000} (\hat{\theta}_i - \theta)^2}, \tag{17}$$

are reported in Tables 1–2. Overall, the parameters can be satisfactorily estimated in all the scenarios, as the RMSEs are all quite small in Table 2. It is observed that the RBs for estimating the 0.05 lifetime quantiles can be relatively large (10%–20%) when $\beta = 0.8$ and $n \le 30$. This is because the true values of the 0.05 lifetime quantiles are small and a small bias on the parameter estimates may yield a large bias in estimating the lifetime quantiles. On the other hand, the estimation performance in terms of RMSE is still satisfactory and when $n = 50$, the RBs of the 0.05 lifetime quantiles are all less than 10%.

We then the performance of the proposed GPQ method in constructing the CIs for the model parameters as well as the lifetime quantiles. We set N=10,000 in Algorithm 1 and use the empirical percentiles for interval estimation with the nominal level 95%. Based on 2,000 replications, the coverage probabilities (CPs) are shown in Table 3, where the other model settings are the same as those of the point estimation procedures. As seen, the CPs are always very close the nominal level regardless of the parameter settings and sample sizes, indicating the superiority of the proposed interval estimation procedures. The almost accurate quantification of the uncertainties evidently provides precise information on the component reliability, which is valuable in designing new systems or comparing competitive designs.

5. Applications

For illustrative purposes, two examples are analysed to show how the proposed framework can be applied in practice. We first revisit the lifetime data of brake pads and then conduct a detailed analysis on a dataset from the UEFA Champion's League. In both applications, three commonly-used forms of $H_\beta(t)$, i.e., t^β, $\exp(\beta t) - 1$ and $\log(1 + t/\beta)$, are used for comparison. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) are used for model selection and the model with the smallest AIC or BIC value is selected.

5.1. Brake pad data analysis

We revisit the lifetime data of the front and rear brake pads as introduced in Section 1. We first check the correlations between the lifetimes of the front and rear brake pads. Unlike the classical Pearson's correlation coefficient that describes the linear relationship between two random variables, Spearman's rank correlation is used to characterize nonlinear correlation. The Spearman's rank correlation for this dataset is $\rho = 0.336$, and the p-value of testing for association between T_1 and T_2 is 0.048, which means that correlation exists between the two random variables under the significance level 0.05. The proposed model (6) with $H_\beta(t) = t^\beta$, $\exp(\beta t) - 1$ and $\log(1 + t/\beta)$ is fitted for comparison. The point estimates and the 95% CIs of the model parameters are obtained for each form of $H_\beta(t)$, as listed in Table 4. The values of AIC and BIC suggest that the model with $H_\beta(t) = t^\beta$ provides the best fit.

Once $H_\beta(t) = t^\beta$ is selected, we could use the ML estimates to estimate the individual reliability of font and rear brake pads. For example, the estimated marginal cumulative distribution functions (CDFs) are plotted in Fig. 2, where the empirical CDFs are also plotted. As seen, the estimated results and the empirical results match reasonably well, indicating the appropriateness of our modelling and estimation framework. The estimated reliability information is valuable in many subsequent decision-making activities including reliability prediction of new car, prediction of warranty costs, planing of maintenance activities, and assessment of the effects of a proposed design change [40,41].

Table 1
The RB of the estimates for different sample sizes and models.

$\beta = 0.8$										
Model	Sample size (n)	Parameters								
		β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$
$d = 1.5$	20	4.836	1.154	5.533	2.613	0.308	16.874	7.102	23.153	0.934
	30	3.562	0.126	4.486	2.243	0.935	12.854	4.475	17.946	1.426
	50	1.431	0.234	2.054	1.389	0.702	5.648	2.197	9.575	0.829
$d = 2$	20	5.154	1.126	5.644	2.737	0.928	15.520	7.214	23.650	1.032
	30	3.582	0.452	4.311	1.931	1.306	12.865	4.431	17.905	1.519
	50	1.634	0.246	2.253	1.312	1.865	4.642	2.867	9.617	0.723
$d = 3$	20	5.207	1.379	5.016	5.684	5.046	15.688	7.351	23.363	1.850
	30	3.583	0.754	3.968	5.137	4.556	12.772	4.396	17.502	1.640
	50	1.694	0.430	1.953	2.102	1.318	5.673	2.378	9.847	0.972
$\beta = 3$										
Model	Sample size (n)	Parameters								
		β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$
$d = 1.5$	20	4.924	1.179	5.754	2.263	0.313	4.491	1.951	6.112	0.245
	30	3.528	0.304	4.291	2.986	0.905	3.497	1.138	3.898	0.476
	50	1.711	0.438	2.151	1.551	0.763	1.534	0.637	2.539	0.218
$d = 2$	20	4.975	1.462	5.557	3.083	0.796	4.477	1.934	6.054	0.284
	30	3.571	0.748	4.105	2.442	1.224	3.565	1.147	3.981	0.538
	50	1.710	0.497	1.983	1.818	1.842	1.517	0.641	2.564	0.241
$d = 3$	20	5.026	1.718	4.917	4.107	5.046	4.388	1.931	5.897	0.371
	30	3.584	0.991	3.798	3.638	4.296	3.605	1.135	4.057	0.561
	50	1.704	0.543	1.844	1.012	1.211	1.566	0.656	2.576	0.272

Table 2
The RMSE of the estimates for different sample sizes and models.

$\beta = 0.8$										
Model	Sample size (n)	Parameters								
	β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$	
$d = 1.5$	20	0.133	0.276	0.224	0.247	1.856	0.023	1.099	0.053	1.776
	30	0.107	0.205	0.181	0.187	1.431	0.018	0.865	0.039	1.479
	50	0.074	0.149	0.128	0.138	0.893	0.012	0.775	0.025	1.101
$d = 2$	20	0.135	0.296	0.216	0.268	2.152	0.022	1.116	0.049	1.741
	30	0.106	0.224	0.185	0.232	1.823	0.016	0.857	0.037	1.457
	50	0.078	0.167	0.130	0.176	1.396	0.011	0.761	0.024	1.003
$d = 3$	20	0.136	0.336	0.229	0.293	2.330	0.018	1.105	0.048	1.663
	30	0.106	0.240	0.181	0.261	1.930	0.013	0.740	0.036	1.423
	50	0.079	0.183	0.132	0.186	1.813	0.009	0.769	0.023	1.026
$\beta = 3$										
Model	Sample size (n)	Parameters								
		β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$
$d = 1.5$	20	0.512	0.280	0.220	0.232	1.925	0.070	0.119	0.097	0.120
	30	0.408	0.204	0.183	0.193	1.425	0.057	0.095	0.079	0.098
	50	0.297	0.156	0.141	0.135	0.940	0.042	0.077	0.060	0.077
$d = 2$	20	0.511	0.309	0.224	0.245	2.375	0.068	0.115	0.095	0.118
	30	0.407	0.223	0.187	0.207	1.985	0.055	0.092	0.078	0.097
	50	0.297	0.171	0.144	0.183	1.524	0.041	0.075	0.059	0.076
$d = 3$	20	0.510	0.351	0.231	0.261	2.822	0.065	0.111	0.093	0.116
	30	0.407	0.251	0.193	0.234	2.515	0.053	0.089	0.076	0.095
	50	0.296	0.192	0.148	0.208	2.413	0.039	0.072	0.057	0.075

5.2. Soccer data analysis

The second example is the UEFA Champion's League (soccer) data from [42]. The data have two variables: the first variable T_1 represents the time in minutes of the first goal of any type scored by the home team, and the second variable T_2 denotes the time of the first kick goal scored by any team. The sample size of the original data is 37. In this case all possibilities are open, for instance $T_1 < T_2$, $T_1 > T_2$ or $T_1 = T_2$, and they occur with positive probabilities. Thus, the joint PDF in (5) can be used to analyse this dataset. All the data points have been divided by 60, so that the unit is changed to be hour and the model parameters are of the same order. Based on the complete data, the AIC and BIC values are computed for $H_\beta(t) = t^\beta$, $\exp(\beta t) - 1$ and $\log(1 + t/\beta)$, respectively. The model with $H_\beta(t) = \exp(\beta t) - 1$ has the smallest AIC and BIC values, which are 91.337 and 97.781, respectively.

For illustrating the effectiveness of model (6), we use the data with $T_1 < T_2$, and only 17 observations are kept. Firstly, we check the correlation between the two variables. For this dataset, the Spearman's rank correlation $\rho = 0.556$, and the p-value of testing for association between T_1 and T_2 is 0.021, which indicates that the two random variables are correlated under the significance level 0.05. Secondly, the proposed model is applied to analyse this dataset. The point and interval estimation of the parameters are obtained by the proposed generalized inference. The results as well as AIC and BIC values are listed in Table 5. From Table 5, we see that the model with $H_\beta(t) = \exp(\beta t) - 1$ has the smallest AIC and BIC, which is the same as the results

Table 3
The CP of the estimates for different sample sizes and models.

$\beta = 0.8$										
Model	Sample size (n)	Parameters								
		β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$
$d = 1.5$	20	0.9545	0.9435	0.9515	0.9605	0.9555	0.9565	0.9445	0.9535	0.9465
	30	0.9470	0.9530	0.9495	0.9590	0.9465	0.9505	0.9490	0.9485	0.9505
	50	0.9515	0.9490	0.9505	0.9550	0.9480	0.9505	0.9505	0.9495	0.9500
$d = 2$	20	0.9535	0.9450	0.9525	0.9615	0.9595	0.9560	0.9470	0.9535	0.9455
	30	0.9470	0.9530	0.9495	0.9595	0.9465	0.9505	0.9490	0.9510	0.9515
	50	0.9485	0.9480	0.9495	0.9505	0.9520	0.9490	0.9520	0.9510	0.9505
$d = 3$	20	0.9530	0.9455	0.9535	0.9540	0.9540	0.9550	0.9475	0.9535	0.9450
	30	0.9470	0.9535	0.9500	0.9520	0.9465	0.9510	0.9500	0.9475	0.9505
	50	0.9485	0.9480	0.9505	0.9495	0.9485	0.9515	0.9495	0.9485	0.9505
$\beta = 3$										
Model	Sample size (n)	Parameters								
		β	λ_1	λ_2	μ	d	$t_{1,0.05}$	$t_{1,0.95}$	$t_{2,0.05}$	$t_{2,0.95}$
$d = 1.5$	20	0.9515	0.9435	0.9525	0.9645	0.9665	0.9585	0.9480	0.9535	0.946
	30	0.9445	0.9460	0.9450	0.9560	0.9570	0.9430	0.9465	0.9440	0.938
	50	0.9445	0.9505	0.9500	0.9455	0.9390	0.9460	0.9420	0.9420	0.955
$d = 2$	20	0.9515	0.9450	0.9520	0.9615	0.9690	0.9590	0.948	0.9535	0.9445
	30	0.9445	0.9485	0.9445	0.9565	0.9585	0.9430	0.947	0.9435	0.9390
	50	0.9445	0.9470	0.9495	0.9505	0.9390	0.9465	0.942	0.9425	0.9545
$d = 3$	20	0.9510	0.9445	0.9545	0.9565	0.9615	0.9590	0.9475	0.953	0.9455
	30	0.9445	0.9490	0.9450	0.9520	0.9480	0.9430	0.9475	0.944	0.9385
	50	0.9445	0.9460	0.9515	0.9485	0.9385	0.9460	0.9425	0.943	0.9565

Table 4
The point and 95% CI estimates of the parameter for the lifetime data of the brake pads.

$H_\beta(t)$	Estimate	β	μ	d	λ_1	$\lambda_2 \times 100$	AIC	BIC
t^β	Point	4.105	0.106	11.506	0.266	0.272		
	2.5%	3.372	0.063	6.381	0.158	0.069	**67.431**	**72.097**
	97.5%	4.928	0.163	19.362	0.407	0.870		
$\exp(\beta t) - 1$	Point	1.067	0.132	15.184	0.325	0.928		
	2.5%	0.384	0.064	6.393	0.159	0.070	83.778	88.444
	97.5%	1.654	0.166	19.586	0.410	1.170		
$\log(1 + t/\beta)$	Point	0.234	0.121	12.557	0.251	0.303		
	2.5%	0.084	0.067	5.273	0.152	0.074	121.120	125.786
	97.5%	0.369	0.172	18.569	0.398	0.883		

based on the complete data. The effects of the dynamic environment is characterized by ED process $\mathcal{ED}(0.54[\exp(1.695t) - 1], 1)$ with $d = 2.297$. Since $d = 2$ (corresponding to gamma process) lies in the 95% CI of d, i.e., (1.329, 2.948), we can state that gamma process reflects the effects of the dynamic environment. Fig. 3 shows the estimated marginal CDFs and their 95% CIs for T_1 and T_2, where we can see that the estimated CDF fits the empirical CDF reasonably well.

6. Conclusion

This paper proposed a general statistical framework for system reliability evaluation considering the dynamic operating environments, the correlated component lifetimes, and the lifetime ordering constraints. To the best of our knowledge, this is the first study which simultaneously tackles the above-mentioned three practical issues. The exponential dispersion process was proposed to model the cumulative hazard function, which effectively captured the effects brought by the dynamic environments and the resulting correlated lifetimes. By truncating the joint lifetime distribution, the ordering constraint was further incorporated. Based on the developed model, the point and interval estimates of the model parameters and the lifetime quantiles were readily obtained. In particular, simulation studies showed that the proposed confidence intervals always maintained a very accurate coverage regardless of the sample sizes and parameter settings. At last, the usefulness and effectiveness of the proposed framework was well illustrated by the real brake pads lifetime data and soccer data.

Future research may focus on systems with more than two components. A direct extension of the proposed framework is not easy as lifetime ordering of multiple components can be complicated and the generalized pivotal quantities cannot be straightforwardly constructed. In addition, it is possible to consider random effects among different systems when the lifetime data are from different sources. One way of doing this is to allow one model parameter to change over systems. However, such a treatment may bring in difficulties in statistical inference. Furthermore, more accurate evaluation of the system reliability is expected if the lifetime data from field and lab test are jointly analysed [43]. We believe this study could serve as the building block for all the mentioned research directions but substantial efforts are still needed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 5
The point and 95% CI estimates of the parameter based on the soccer data with $T_1 < T_2$.

$H_\beta(t)$	Estimate	β	μ	d	λ_1	λ_2	AIC	BIC
	Point	1.567	8.351	11.697	3.450	2.085		
t^β	2.5%	1.040	2.051	3.634	1.893	1.213	11.00	13.50
	97.5%	2.164	12.223	19.818	5.542	3.223		
	Point	1.694	0.540	2.297	0.783	0.449		
$\exp(\beta t)-1$	2.5%	1.045	0.055	1.329	0.468	0.203	**8.20**	**10.70**
	97.5%	2.147	1.154	2.948	1.475	0.867		
	Point	1.673	3.156	2.214	3.289	1.886		
$\log(1+t/\beta)$	2.5%	1.142	2.042	1.351	1.788	1.222	22.2	24.70
	97.5%	2.679	4.861	3.460	5.312	3.151		

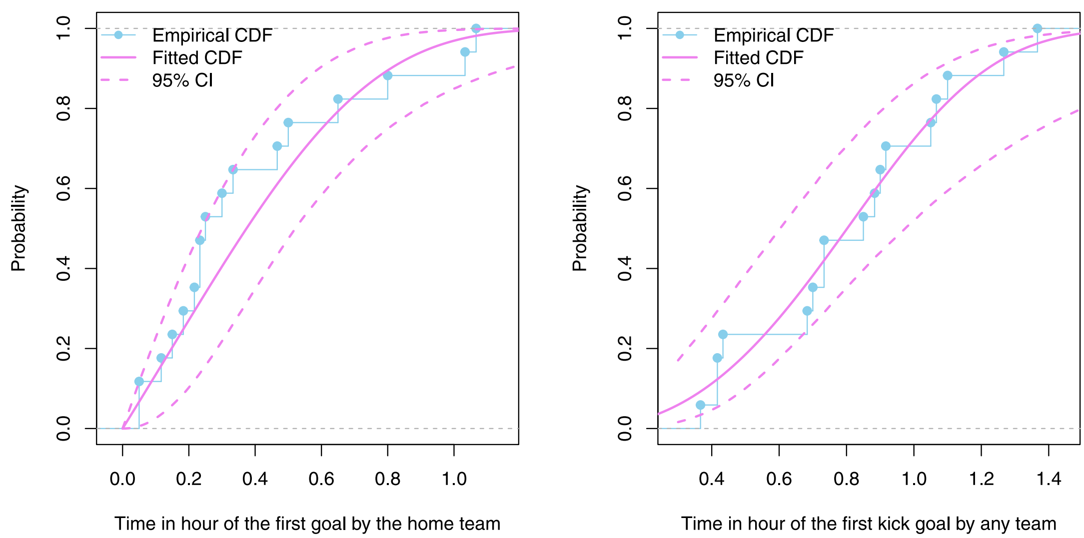

Fig. 3. Fitted marginal CDFs of T_1 and T_2 based on soccer data with $T_1 < T_2$.

Appendix

A.1. Derivations of joint distribution of T_1 and T_2 under model (4)

(i) According to the property of ED process, we know that $C(t_{(1)})$ and $\Delta C(t_{(2)}) = C(t_{(2)}) - C(t_{(1)})$ are independent. $C(t_{(1)}) \sim ED(\mu H_\beta(t_{(1)}), \eta)$ and $\Delta C(t_{(2)}) \sim ED(\mu \Delta H_\beta(t_{(2)}), \eta)$. Conditioned on $C(t_{(1)})$ and $\Delta C(t_{(2)})$,

$$P(T_1 > t_1, T_2 > t_2 | C(t_{(1)}), \Delta C(t_{(2)})) = \exp\{-2C(t_{(1)})\}\exp\{-\Delta C(t_{(2)})\}.$$

Then taking the expectation of $P(T_1 > t_1, T_2 > t_2 | C(t_{(1)}), \Delta C(t_{(2)}))$ with respect to $C(t_{(1)})$ and $\Delta C(t_{(2)})$, we have

$$\begin{aligned} S(t_1,t_2) &= P(T_1 > t_1, T_2 > t_2) \\ &= \mathrm{E}_{C(t_{(1)})}[\exp\{-2C(t_{(1)})\}]\mathrm{E}_{\Delta C(t_{(2)})}[\exp\{-\Delta C(t_{(2)})\}] \\ &= \exp\{\eta H_\beta(t_{(1)})[\kappa(\theta - 2/\eta) - \kappa(\theta)]\} \\ &\quad \times \exp\{\eta \Delta H_\beta(t_{(2)})[\kappa(\theta - 1/\eta) - \kappa(\theta)]\} \\ &= \exp\left\{-\eta H_\beta(t_{(1)})\left[\kappa(\theta - 1/\eta) - \kappa(\theta - 2/\eta)\right] - \eta H_\beta(t_{(2)}) \right. \\ &\quad \left. \times \left[\kappa(\theta) - \kappa(\theta - 1/\eta)\right]\right\}, \end{aligned}$$

where the calculation of the third equality is from (3).

(ii) Two random variables T_1 and T_2 are said to be positive quadrant dependent if

$$P(T_1 > t_1, T_2 > t_2) \geq P(T_1 > t_1)P(T_2 > t_2),$$

for all t_1 and t_2.

The survival function of T_i can be derived as

$$\begin{aligned} P(T_i > t_i) &= \mathrm{E}_{C(t_i)}[\exp\{-C(t_i)\}] \\ &= \exp\{-\eta H_\beta(t_i)[\kappa(\theta) - \kappa(\theta - 1/\eta)]\}, \ i = 1, 2. \end{aligned}$$

Then we have

$$\begin{aligned} Q(t_1,t_2) &= \frac{P(T_1 > t_1, T_2 > t_2)}{P(T_1 > t_1)P(T_2 > t_2)} \\ &= \exp\{\eta H_\beta(t_{(1)})[(\kappa(\theta) - \kappa(\theta - 1/\eta)) \\ &\quad - (\kappa(\theta - 1/\eta) - \kappa(\theta - 2/\eta))]\}. \end{aligned} \tag{18}$$

Since $\kappa(\cdot)$ is a convex function, we have that $[\kappa(\theta) - \kappa(\theta - 1/\eta)] - [\kappa(\theta - 1/\eta) - \kappa(\theta - 2/\eta)] \geq 0$. Then $Q(t_1, t_2) \geq 1$. Therefore, T_1 and T_2 are positive quadrant dependent.

A.2. Derivations of joint distribution of T_1 and T_2 under $T_1 < T_2$

First, we compute the probability of $T_1 < T_2$, that is,

$$\begin{aligned} P(T_1 < T_2) &= \int_0^\infty \int_0^{t_2} f(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 \\ &= \int_0^\infty \int_0^{t_2} \eta^2 \nabla\kappa_1 \nabla\kappa_2 \prod_{i=1}^{2} \frac{\partial H_\beta(t_i)}{\partial t_i} \\ &\quad \times \exp\{-\eta H_\beta(t_1)\nabla\kappa_1 - \eta H_\beta(t_2)\nabla\kappa_2\}\mathrm{d}t_1\mathrm{d}t_2 \\ &= \int_0^\infty \eta \nabla\kappa_2 \frac{\partial H_\beta(t_2)}{\partial t_2} \exp\{-\eta H_\beta(t_2)\nabla\kappa_2\} \\ &\quad \times [1 - \exp\{-\eta H_\beta(t_2)\nabla\kappa_1\}]\mathrm{d}t_2 \\ &= 1 - \frac{\nabla\kappa_2}{\nabla\kappa_1 + \nabla\kappa_2} = \frac{\nabla\kappa_1}{\nabla\kappa_0}. \end{aligned}$$

Thus, the joint PDF of T_1 and T_2 on the region $T_1 < T_2$ is

$$f_{TR}(t_1,t_2) = \frac{f(t_1,t_2)}{P(T_1 < T_2)} = \eta^2 \nabla\kappa_0 \nabla\kappa_2 \prod_{i=1}^{2} \frac{\partial H_\beta(t_i)}{\partial t_i} \times \exp\{-\eta H_\beta(t_1)\nabla\kappa_1 - \eta H_\beta(t_2)\nabla\kappa_2\}, t_1 < t_2.$$

A.3. Order of total positivity between T_1 and T_2

Here we show that T_1 and T_2 are totally positive of order 2. We first introduce the definition given by [35].

Definition 1. Let X and Y have a joint PDF $p(x,y)$. Then $p(x,y)$ is said to be totally positive of order 2 if for all $x_1 < x_2$, $y_1 < y_2$, $p(x_1,y_1)p(x_2,y_2) \geq p(x_1,y_2)p(x_2,y_1)$.

From Definition 1, we should prove that for all $t_{11} < t_{12}$, $t_{21} < t_{22}$,

$$f_{TR}\left(t_{11},t_{21}\right) f_{TR}\left(t_{12},t_{22}\right) \geq f_{TR}\left(t_{11},t_{22}\right) f_{TR}\left(t_{12},t_{21}\right) \quad (19)$$

where t_{i1} and t_{i2} are any observed values of T_i. We discuss the results

1. If $t_{12} \geq t_{21}$, then $f_{TR}(t_{12},t_{21}) = 0$. The inequality (19) holds.
2. If $t_{12} < t_{21}$, then

$$f_{TR}\left(t_{11},t_{21}\right) f_{TR}\left(t_{12},t_{22}\right) = f_{TR}\left(t_{11},t_{22}\right) f_{TR}\left(t_{12},t_{21}\right) = \eta^4 \left[\nabla\kappa_0\right]^2 \left[\nabla\kappa_2\right]^2 \prod_{i=1}^{2}\prod_{j=1}^{2} \frac{\partial H_\beta\left(t_{ij}\right)}{\partial t_{ij}} \times \exp\left\{-\eta\left[H_\beta\left(t_{11}\right) + H_\beta\left(t_{12}\right)\right] \times \nabla\kappa_1 - \eta\left[H_\beta\left(t_{21}\right) + H_\beta\left(t_{22}\right)\right]\nabla\kappa_2\right\}.$$

Thus, the results hold.

A.4. Proof of Theorem 1

The transformation is one-to-one respondence, and we have $T_1 = H_\beta^{-1}(Z_1), T_2 = H_\beta^{-1}(Z_1 + Z_2)$. Let $\mathcal{J}$ be the Jacobian matrix of the transformation from $(Z_1, Z_2)'$to $(T_1, T_2)'$, and

$$\mathcal{J} = \left|\frac{\partial(T_1,T_2)'}{\partial(Z_1,Z_2)'}\right| = \frac{1}{H'_\beta(H_\beta^{-1}(Z_1)) \cdot H'_\beta(H_\beta^{-1}(Z_1+Z_2))}.$$

Thus, the joint PDF of $Z = (Z_1, Z_2)'$ is

$$f_Z(z_1,z_2) = f_{TR}(t_1,t_2)\cdot\mathcal{J} = \eta^2\nabla\kappa_0\nabla\kappa_2 \exp\{-\eta\nabla\kappa_1 z_1 - \eta\nabla\kappa_2(z_1+z_2)\} = \eta\nabla\kappa_0 \exp\{-\eta\nabla\kappa_0 z_1\}\cdot\eta\nabla\kappa_2\exp\{-\eta\nabla\kappa_2 z_2\} = (\lambda_1+\lambda_2)\exp\{-(\lambda_1+\lambda_2)z_1\}\cdot\lambda_2\exp\{-\lambda_2 z_2\}.$$

Thus, the results hold.

References

[1] Ding L, Wang H, Jiang J, Xu A. SIL verification for SRS with diverse redundancy based on system degradation using reliability block diagram. Reliab Eng Syst Saf 2017;165:170–87.
[2] Mi J, Beer M, Li Y-F, Broggi M, Cheng Y. Reliability and importance analysis of uncertain system with common cause failures based on survival signature. Reliab Eng Syst Saf 2020;201:106988.
[3] Fiondella L, Rajasekaran S, Gokhale SS. Efficient software reliability analysis with correlated component failures. IEEE Trans Reliab 2013;62(1):244–55.
[4] Pezoa JE, Hayat MM. Reliability of heterogeneous distributed computing systems in the presence of correlated failures. IEEE Trans Parallel Distrib Syst 2013;25(4):1034–43.
[5] Zhang Y, Yin E, Li F, Zhang Y, Tanaka T, Zhao Q, Cui Y, Xu P, Yao D, Guo D. Two-stage frequency recognition method based on correlated component analysis for SSVEP-based BCI. IEEE Trans Neural Syst Rehabil Eng 2018;26(7):1314–23.
[6] Lin Y-K, Chang P-C, Fiondella L. Quantifying the impact of correlated failures on stochastic flow network reliability. IEEE Trans Reliab 2012;61(3):692–701.
[7] Yalaoui A, Chu C, Chatelet E. Reliability allocation problem in a series–parallel system. Reliab Eng Syst Saf 2005;90(1):55–61.
[8] Tekiner-Mogulkoc H, Coit DW. System reliability optimization considering uncertainty: Minimization of the coefficient of variation for series-parallel systems. IEEE Trans Reliab 2011;60(3):667–74.
[9] Feizollahi MJ, Ahmed S, Modarres M. The robust redundancy allocation problem in series-parallel systems with budgeted uncertainty. IEEE Trans Reliab 2014;63(1):239–50.
[10] Fan T-H, Hsu T-M. Statistical inference of a two-component series system with correlated log-normal lifetime distribution under multiple type-I censoring. IEEE Trans Reliab 2014;64(1):376–85.
[11] Fan T-H, Wang Y-F, Ju S-K. A competing risks model with multiply censored reliability data under multivariate Weibull distributions. IEEE Trans Reliab 2019;68(2):462–75.
[12] Lemonte AJ. Multivariate Birnbaum–Saunders regression model. J Stat Comput Simul 2013;83(12):2244–57.
[13] Liu Y, Liu Q, Xie C, Wei F. Reliability assessment for multi-state systems with state transition dependency. Reliab Eng Syst Saf 2019;188:276–88.
[14] Sun F, Fu F, Liao H, Xu D. Analysis of multivariate dependent accelerated degradation data using a random-effect general Wiener process and D-vine copula. Reliab Eng Syst Saf 2020;204:107168.
[15] Lamboni M, Kucherenko S. Multivariate sensitivity analysis and derivative-based global sensitivity measures with dependent variables. Reliab Eng Syst Saf 2021;212:107519.
[16] Xu A, Zhou S, Tang Y. A unified model for system reliability evaluation under dynamic operating conditions. IEEE Trans Reliab 2021;70(1):65–72.
[17] Okaro IA, Tao L. Reliability analysis and optimisation of subsea compression system facing operational covariate stresses. Reliab Eng Syst Saf 2016;156:159–74.
[18] Izquierdo J, Márquez AC, Uribetxebarria J. Dynamic artificial neural network-based reliability considering operational context of assets. Reliab Eng Syst Saf 2019;188:483–93.
[19] Li Y-F, Liu Y, Huang T, Huang H-Z, Mi J. Reliability assessment for systems suffering common cause failure based on Bayesian networks and proportional hazards model. Qual Reliab Eng Int 2020;36(7):2509–20.
[20] Singpurwalla ND, Youngren MA. Multivariate distributions induced by dynamic environments. Scand J Stat 1993;20(3):251–61.
[21] Zheng W, Liyang X. Dynamic reliability model of components under random load. IEEE Trans Reliab 2008;57(3):474–9.
[22] Khakzad N, Khan F, Amyotte P. Dynamic risk analysis using bow-tie approach. Reliab Eng Syst Saf 2012;104:36–44.
[23] Hong L, Zhai Q, Wang X, Ye Z. System reliability evaluation under dynamic operating conditions. IEEE Trans Reliab 2019;68(3):800–9.
[24] Eryilmaz S. The effectiveness of adding cold standby redundancy to a coherent system at system and component levels. Reliab Eng Syst Saf 2017;165:331–5.
[25] Dui H, Si S, Yam RC. Importance measures for optimal structure in linear consecutive-k-out-of-n systems. Reliab Eng Syst Saf 2018;169:339–50.
[26] Zhang Y. Reliability analysis of randomly weighted k-out-of-n systems with heterogeneous components. Reliab Eng Syst Saf 2021;205:107184.
[27] Ramírez-Márquez JE, Jiang W. On improved confidence bounds for system reliability. IEEE Trans Reliab 2006;55(1):26–36.
[28] Guo H, Jin T, Mettas A. Designing reliability demonstration tests for one-shot systems under zero component failures. IEEE Trans Reliab 2010;60(1):286–94.
[29] Xiao M, Zhang J, Gao L. A system active learning Kriging method for system reliability-based design optimization with a multiple response model. Reliab Eng Syst Saf 2020;199:106935.
[30] Tseng ST, Lee I. Optimum allocation rule for accelerated degradation tests with a class of exponential-dispersion degradation models. Technometrics 2016;58(2):244–54.
[31] Zhou S, Xu A. Exponential dispersion process for degradation analysis. IEEE Trans Reliab 2019;68(2):398–409.
[32] Chen Z, Pan E, Xia T, Li Y. Optimal degradation-based burn-in policy using tweedie exponential dispersion process model with measurement errors. Reliab Eng Syst Saf 2020;195:106748.
[33] Dunn P, Smyth G. Series evaluation of tweedie exponential dispersion model densities. Stat Comput 2005;15(4):267–80.
[34] Kotz S, Lai C, Xie M. On the effect of redundancy for systems with dependent components. IIE Trans 2003;35(12):1103–10.
[35] Shaked M. Concept of positive dependence for exchangeable random variables. Ann Statist 1977;5(3):505–15.
[36] Lai C, Xie M. Stochastic ageing and dependence for reliability. Springer; 2006.
[37] Wang B, Yu K, Jones M. Inference under progressively type II right-censored sampling for certain lifetime distributions. Technometrics 2010;52(4):453–60.
[38] Casella G, Berger RL. Statistical inference. Cengage Learning; 2021.
[39] Weerahandi S. Generalized confidence intervals. J Amer Statist Assoc 1993;88:899–905.
[40] Chen P, Ye Z-S. Estimation of field reliability based on aggregate lifetime data. Technometrics 2017;59(1):115–25.
[41] Hu J, Chen P. Predictive maintenance of systems subject to hard failure based on proportional hazards model. Reliab Eng Syst Saf 2020;196:106707.
[42] Meintanis S. Test of fit for Marshall–Olkin distributions with applications. J Statist Plann Inference 2007;137:3954–63.
[43] Ye Z-S, Hong Y, Xie Y. How do heterogeneities in operating environments affect field failure predictions and test planning? Ann Appl Stat 2013;7(4):2249–71.