Sparse Phase Retrieval Via PhaseLiftOff

Yu Xia and Zhiqiang Xu

***Abstract*—The aim of sparse phase retrieval is to recover a k-sparse signal $\mathbf{x}_0 \in \mathbb{C}^d$ from quadratic measurements $|\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2$ where $\mathbf{a}_j \in \mathbb{C}^d, j = 1, \ldots, m$. Noting $|\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2 = \mathrm{Tr}(A_j X_0)$ with $A_j = \mathbf{a}_j\mathbf{a}_j^* \in \mathbb{C}^{d\times d}$, $X_0 = \mathbf{x}_0\mathbf{x}_0^* \in \mathbb{C}^{d\times d}$, one can recast sparse phase retrieval as a problem of recovering a rank-one sparse matrix from linear measurements. Yin and Xin introduced *PhaseLiftOff* which presents a proxy of rank-one condition via the difference of trace and Frobenius norm. By adding sparsity penalty to *PhaseLiftOff*, in this paper, we present a novel model to recover sparse signals from quadratic measurements. Theoretical analysis shows that the optimal solution to our model provides the stable recovery of $\mathbf{x}_0$ under almost optimal sampling complexity $m = O(k \log(d/k))$. We use the difference of convex function algorithm (DCA) to solve PhaseLiftOff, showing DCA converges to a stationary point. Numerical experiments demonstrate that our algorithm outperforms other state-of-the-art algorithms used for solving sparse phase retrieval.**

***Index Terms*—Signal recovery, phase retrieval, compressed sensing, restricted isometry property, compressed phaseless sensing.**

I. Introduction

A. Phase Retrieval

PHASE retrieval is raised in many areas, such as X-ray crystallography, astronomy, quantum tomography, optics and microscopy. We assume that $\mathbf{x}_0 \in \mathbb{F}^d$ is a target signal, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The aim of phase retrieval is to recover $\mathbf{x}_0 \in \mathbb{F}^d$ from $|\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2 + w_j, j = 1, \ldots, m$, up to a unimodular constant where $\mathbf{a}_j \in \mathbb{F}^d$ are known measurement vectors and $\mathbf{w} := (w_1, \ldots, w_m)^T \in \mathbb{R}^m$ is a noise vector. For convenience, let $\mathcal{A} : \mathbb{F}^{d\times d} \to \mathbb{R}^m$ be a linear map which is defined as

$$\mathcal{A}(X) = (\mathbf{a}_1^* X \mathbf{a}_1, \ldots, \mathbf{a}_m^* X \mathbf{a}_m), \quad (1)$$

where $X \in \mathbb{F}^{d\times d}$, $\mathbf{a}_j \in \mathbb{F}^d, j = 1, \ldots, m$. We abuse the notation and set

$$\mathcal{A}(\mathbf{x}) := \mathcal{A}(\mathbf{x}\mathbf{x}^*) = (|\langle \mathbf{a}_1, \mathbf{x}\rangle|^2, \ldots, |\langle \mathbf{a}_m, \mathbf{x}\rangle|^2),$$

where $\mathbf{x} \in \mathbb{F}^d$. With these notations, we can formulate the aim of phase retrieval as follows: *To estimate the matrix* $X_0 = \mathbf{x}_0\mathbf{x}_0^* \in \mathbb{C}^{d\times d}$ *from* $\mathcal{A}(\mathbf{x}_0) + \mathbf{w} \in \mathbb{R}^m$.

For the noiseless case, to guarantee the solution $\mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{x}_0)$ is unique for all $\mathbf{x}_0 \in \mathbb{C}^d$, it is shown in [27] that the measurement number $m \geq 4d - 2 - 2\alpha_d$ is necessary where α_d denotes the number of 1's in the binary of expansion of $d - 1$. The authors in [9] proved that $m \geq 4d - 4$ generic measurement vectors $\mathbf{a}_j \in \mathbb{C}^d, j = 1, \ldots, m$, are enough to guarantee the uniqueness of the solution.

In [5], [7], [8], the phase retrieval was recasted as a semi-definite programming problem, i.e., the PhaseLift problem:

$$\min_{X\in\mathbb{F}^{d\times d}} \mathrm{Tr}(X) \quad \text{s.t.} \quad \mathcal{A}(X) = \mathcal{A}(X_0),\ X \succeq 0. \quad (2)$$

In [7], it is shown that the solution to (2) is X_0 with high probability provided $\mathbf{a}_j$ is Gaussian random vector and $m = O(d \log d)$, which was reduced to $m = O(d)$ in [5]. For the aim of computation, the regularized trace-norm minimization is suggested in [7], [8]:

$$\min_{X\succeq 0, X\in\mathbb{F}^{d\times d}} \frac{1}{2}\|\mathcal{A}(X) - \mathbf{b}\|_2^2 + \lambda \mathrm{Tr}(X). \quad (3)$$

Noting that $\mathrm{Tr}(X) - \|X\|_F \geq 0$ and the equality holds iff $\mathrm{rank}(X) = 1$, Yin and Xin suggested the following regularization problem [30], which is called as *PhaseLiftOff*:

$$\min_{X\succeq 0, X\in\mathbb{F}^{d\times d}} \frac{1}{2}\|\mathcal{A}(X) - \mathbf{b}\|_2^2 + \lambda(\mathrm{Tr}(X) - \|X\|_F). \quad (4)$$

The numerical experiments in [30] showed that PhaseLiftOff outperforms PhaseLift.

B. Sparse Phase Retrieval

In many areas, one also requires $\|\mathbf{x}_0\|_0 \leq k$, i.e., the number of nonzero entries of $\mathbf{x}_0$ less than or equal to k [18], [19], [26]. The aim of *sparse phase retrieval* is to recover the k-sparse signal $\mathbf{x}_0$ from $|\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2 = b_j, j = 1, \ldots, m$.

For convenience, we set $\Sigma_k^{\mathbb{F}} := \{\mathbf{x} \in \mathbb{F}^d : \|\mathbf{x}\|_0 \leq k\}$. It was shown in [26] that, for $\mathbb{F} = \mathbb{C}$ and $\mathbf{x}_0 \in \Sigma_k^{\mathbb{F}}$, if $m \geq 4k - 2$ (resp. $m \geq 2k$ for $\mathbb{F} = \mathbb{R}$) and $\mathbf{a}_1, \ldots, \mathbf{a}_m$ are generic vectors in $\mathbb{C}^d$ (resp. $\mathbb{R}^d$) then the solution to $\mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{x}_0)$ with $\mathbf{x} \in \Sigma_k^{\mathbb{F}}$ is unique up to a unimodular constant.

The ℓ_1-minimization is a commonly used method for recovering sparse signals. Naturally, one is also interested in employing ℓ_1-minimization for solving sparse phase retrieval. For $\mathbb{F} = \mathbb{R}$, the following model was considered in [24]:

$$\min_{\mathbf{x}\in\mathbb{R}^{d\times d}} \|\mathbf{x}\|_1 \quad \text{s.t.} \quad \mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{x}_0). \quad (5)$$

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Particularly, it is proved that the solution to (5) is $\pm\mathbf{x}_0$ with high probability if $m \gtrsim k \log d$ and $\mathbf{a}_j, j = 1, \ldots, m$, are independent Gaussian random vectors. In [29], the authors extended this result to the case where $\mathbb{F} = \mathbb{C}$.

In [15], the following convex model was considered

$$\min_{X \in \mathbb{R}^{d\times d}} \|X\|_1 + \lambda \mathrm{Tr}(X), \quad \text{s.t.} \quad \mathcal{A}(X) = \mathcal{A}(\mathbf{x}_0),\ X \succeq 0. \tag{6}$$

The objective function in (6) is the summation of the trace and the ℓ_1 norm, which is also a convex model. To guarantee the solution to (6) is $\mathbf{x}_0\mathbf{x}_0^*$, one has to require the number of measurements $m \gtrsim k^2 \log d$, which is quadratic about the sparse level k [15].

Beyond the convex model, one also develops many nonconvex algorithms for solving sparse phase retrieval, such as Sparse Truncated Amplitude flow (SPARTA) [25], Thresholded Wirtinger Flow (ThWF) [4], Sparse Wirtinger Flow (SWF) [32], Sparse Phase Retrieval via Smoothing Function (SPRSF) [17], Compressive Phase Retrieval with Alternating Minimization (CoPRAM) [13]. These algorithms include two stages: (i) Recover the support of the underlying sparse signal under some analytical rule, and construct an initialization near the ground truth signal $\mathbf{x}_0$; (ii) Refine the initialization by gradient-type iterations and extra truncation procedure by hard thresholding. However, to guarantee the algorithms converge to the true signal, the algorithms mentioned above require the sample complexity is $m = O(k^2 \log d)$.

C. Our Contribution

A natural model for sparse phase retrieval is to use ℓ_1-regularization methods, i.e.,

$$\begin{aligned} \min_{X \in \mathbb{C}^{d\times d}} \ & \mu\|X\|_1 + \frac{1}{2}\|\mathcal{A}(X) - \mathbf{b}\|_2^2, \\ & \text{s.t. } X \succeq 0,\ \mathrm{rank}(X) = 1, \end{aligned} \tag{7}$$

where $\|X\|_1 = \sum_{l=1}^d \sum_{j=1}^d |x_{j,l}|$ and $x_{j,l}$ are the entries of X. Motivated by the notable PhaseLiftOff [30], we reformulate (7) as the following regularization problem:

$$\begin{aligned} \min_{X \in \mathbb{C}^{d\times d}} \ & \lambda(\mathrm{Tr}(X) - \|X\|_F) + \mu\|X\|_1 + \frac{1}{2}\|\mathcal{A}(X) - \mathbf{b}\|_2^2 \\ & \text{s.t. } X \succeq 0. \end{aligned} \tag{8}$$

For convenience, we call (8) *Sparse PhaseLiftOff* model. Note that the object function in (8) is the difference of convex functions and hence it can be solved by the *difference of convex functions algorithm* (DCA).

To study the performance of (8), we first establish the equivalence between (8) and (7) under some mild conditions about λ, μ and $\|\mathbf{w}\|_2$:

Lemma I.1: Assume that $\mathbf{b} = \mathcal{A}(\mathbf{x}_0\mathbf{x}_0^*) + \mathbf{w}$ where $\mathbf{x}_0 \in \mathbb{C}^d$, and $\mathbf{w} \in \mathbb{R}^m$ is the noise term. Let $X^\#$ be the global minimizer of (8). If

$$\frac{1}{2}\|\mathbf{b}\|_2^2 > \mu\|\mathbf{x}_0\|_1^2 + \frac{1}{2}\|\mathbf{w}\|_2^2,\ \mu \geq 0$$

and

$$\lambda > \frac{\mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1 + \|\mathbf{w}\|_2)}{\sqrt{2} - 1}, \tag{9}$$

then $\mathrm{rank}(X^\#) = 1$.

As said before, $\mathrm{Tr}(X) - \|X\|_F = 0$ provided $\mathrm{rank}(X) = 1$. Under the conditions of Lemma I.1, $X^\#$ is also the minimizer of (7). Hence, we turn to study the performance of (7). To do that, we require $\mathcal{A}$ satisfies restricted isometry property over low-rank and sparse matrices:

Definition I.1: [29] We say that the map $\mathcal{A} : \mathbb{H}^{d\times d} \to \mathbb{R}^m$ satisfies the restricted isometry property of order (r, s) if there exist positive constants c and C such that the inequality

$$c\|X\|_F \leq \frac{1}{m}\|\mathcal{A}(X)\|_1 \leq C\|X\|_F \tag{10}$$

holds for all $X \in \mathbb{H}^{d\times d}$ with $\mathrm{rank(X)} \leq \mathrm{r}$ and $\|X\|_{0,2} \leq s$.

Based on the RIP, we have the following theorem.

Theorem I.2: Let $\mathbf{b} = \mathcal{A}(\mathbf{x}_0\mathbf{x}_0^*) + \mathbf{w}$, where $\mathbf{x}_0 \in \mathbb{C}^d$, and $\mathbf{w} \in \mathbb{R}^m$ is the noise term. Assume that $\mathcal{A}(\cdot)$ satisfy the RIP condition of order $(2, 2ak)$ with RIP constant $c, C > 0$, and $a \geq 1$ with

$$c - \frac{4\sqrt{2}C}{\sqrt{a}} - \frac{C}{a} > 0. \tag{11}$$

Set $\alpha := \frac{1 + \frac{1}{a} + \frac{4\sqrt{2}}{\sqrt{a}}}{c - \frac{C}{a} - \frac{4\sqrt{2}C}{\sqrt{a}}}$. Assume that $\mu > 0$. For any k-sparse signals $\mathbf{x}_0 \in \mathbb{C}^d$, the solution to (7) $X^\# := \mathbf{x}^\#(\mathbf{x}^\#)^*$ satisfies

$$\begin{aligned} & \|\mathbf{x}^\#(\mathbf{x}^\#)^* - \mathbf{x}_0\mathbf{x}_0^*\|_F \\ & \leq (2C\alpha + 2)\frac{\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\left(\|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{\mu}}\right) \\ & \quad + \frac{\|\mathbf{w}\|_2^2}{2\mu a k} + \alpha \cdot \frac{\mu a k}{2\,C}\left(\frac{C\|\mathbf{w}\|_2}{\mu a k} + \frac{1}{\sqrt{m}}\right)^2. \end{aligned} \tag{12}$$

The following theorem shows that complex Gaussian random quadratic map $\mathcal{A}$ satisfies RIP of order $(2, k)$ with high probability provided $m \gtrsim k \log(d/k)$.

Theorem I.3: [29] Assume that the linear measurement $\mathcal{A}(\cdot)$ is defined as

$$\mathcal{A}(X) = (\mathbf{a}_1^* X \mathbf{a}_1, \ldots, \mathbf{a}_m^* X \mathbf{a}_m),$$

where $\mathbf{a}_j$ are independently complex Gaussian random vectors, i.e., $\mathbf{a}_j \sim \mathcal{N}(0, \frac{1}{2}\mathbf{I}_{n\times n}) + \mathcal{N}(0, \frac{1}{2}\mathbf{I}_{n\times n})i$. If

$$m \gtrsim k \log(d/k),$$

under the probability at least $1 - 2\exp(-c_0\, m)$, the linear map $\mathcal{A}$ satisfies the restricted isometry property of order $(2, k)$, i.e.

$$0.12\|X\|_F \leq \frac{1}{m}\|\mathcal{A}(X)\|_1 \leq 2.45\|X\|_F,$$

for all $X \in \mathbb{H}^{n\times n}$ with $\mathrm{rank(X)} \leq \mathrm{k}$ and $\|X\|_{0,2} \leq k$ (also $\|X^*\|_{0,2} \leq k$).

Combing Lemma I.1, Theorem I.2 and Theorem I.3, we have the following corollary:

Corollary I.4: Assume that $\mathbf{a}_j, j = 1, \ldots, m$ are independently complex Gaussian random vectors, i.e., $\mathbf{a}_j \sim$

$\mathcal{N}(0, \frac{1}{2}\mathbf{I}_d) + \mathcal{N}(0, \frac{1}{2}\mathbf{I}_d)i$. Assume that $\mathbf{b} = \mathcal{A}(\mathbf{x}_0\mathbf{x}_0^*) + \mathbf{w}$, where $\mathbf{w} \in \mathbb{R}^m$ is a noise vector and $\mathbf{x}_0 \in \mathbb{C}^d$ with $\|\mathbf{x}_0\|_0 \le k$. Assume that $m \gtrsim k\log(ed/k)$. Let $X^\#$ be the global minimizer of model (8). The following holds with probability at least $1 - \exp(-cm)$: If

$$\frac{1}{2}\|\mathbf{b}\|_2^2 > \mu\|\mathbf{x}_0\|_1^2 + \frac{1}{2}\|\mathbf{w}\|_2^2,\ \mu > 0$$

and

$$\lambda > \frac{\mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1 + \|\mathbf{w}\|_2)}{\sqrt{2}-1},$$

then $\operatorname{rank}(X^\#) = 1$, and $X^\# = \mathbf{x}^\#(\mathbf{x}^\#)^*$ satisfies

$$\begin{aligned}&\|\mathbf{x}^\#(\mathbf{x}^\#)^* - \mathbf{x}_0\mathbf{x}_0^*\|_F \\ &\lesssim \frac{\|\mathbf{w}\|_2}{\sqrt{\mu k}}\left(\|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{\mu}}\right) \\ &\quad + \frac{\|\mathbf{w}\|_2^2}{2\mu k} + \mu k\left(\frac{\|\mathbf{w}\|_2}{\mu k} + \frac{1}{\sqrt{m}}\right)^2.\end{aligned}$$

Remark I.5: According to Corollary I.4, the parameter λ depends on $\|\mathcal{A}\|$. We next show $\|\mathcal{A}\| = O((m+d)\sqrt{d})$. We assume that the singular value decomposition of $X \in \mathbb{C}^{d\times d}$ with $\|X\|_F = 1$ is $X = \sum_{j=1}^d \sigma_j \mathbf{u}_j\mathbf{v}_j^*$. Here, $\sum_{j=1}^d \sigma_j^2 = 1$. We claim that $\|\mathcal{A}(\mathbf{u}_j\mathbf{v}_j^*)\|_1 = O(m+d)$ holds with probability at least $1 - \exp(-m)$. Then we have

$$\begin{aligned}\|\mathcal{A}(X)\|_2 \le \|\mathcal{A}(X)\|_1 &\le \sum_j \sigma_j \|\mathcal{A}(\mathbf{u}_j\mathbf{v}_j^*)\|_1 \\ &= O((m+d)\sqrt{d}).\end{aligned}$$

Note that

$$\begin{aligned}\|\mathcal{A}(\mathbf{u}_j\mathbf{v}_j^*)\|_1 &= \sum_i |\langle \mathbf{a}_i, \mathbf{u}_j\rangle\langle \mathbf{a}_i, \mathbf{v}_j\rangle| \\ &\le \sqrt{\sum_j |\langle \mathbf{a}_i, \mathbf{u}_j\rangle|^2}\sqrt{\sum_j |\langle \mathbf{a}_i, \mathbf{v}_j\rangle|^2} \le (\sqrt{m} + \sqrt{d} + t)^2\end{aligned}$$

holds with probability larger than $1 - \exp(-t^2/2)$. Here, the last inequality follows from the singular values of Gaussian random matrices [23, Corrollary 5.35]. By taking $t = \sqrt{m} + \sqrt{d}$, we have $\|\mathcal{A}(\mathbf{u}_j\mathbf{v}_j^*)\|_1 = O(m+d)$.

Remark I.6: If we take $\|\mathbf{w}\|_2 = 0$ in Corollary I.4 then the following holds with high probability:

$$\|\mathbf{x}^\#(\mathbf{x}^\#)^* - \mathbf{x}_0\mathbf{x}_0^*\|_2 \lesssim \frac{\mu k}{m}$$

provided

$$\lambda > \frac{\mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1)}{\sqrt{2}-1}.$$

D. Notations

We use $\mathbb{H}^{d\times d}$ to denote the set of all $d \times d$ Hermitian matrices. For any $X, Y \in \mathbb{H}^{d\times d}$, set $\langle X, Y\rangle := \operatorname{Tr}(X^*Y)$. For $x \in \mathbb{C}$, we use $\mathscr{R}(x)$ and $\mathscr{I}(x)$ to denote the real and complex parts of x, respectively. For $X \in \mathbb{C}^{d\times d}$, we use $X_{i,:}$ and $X_{:,j}$ to denote the i-th row and j-th column of X, respectively. For $S, T \subset \{1, \dots, d\}$, we use $X_{S,T}$ to denote a submatrix of X with the rows indexed in S and columns indexed in T. We also set $\|X\|_1 := \sum_{i,j}\sqrt{\mathscr{R}(X_{i,j})^2 + \mathscr{I}(X_{i,j})^2}$, $\|X\|_F := \sqrt{\sum_{i,j}(\mathscr{R}(X_{i,j})^2 + \mathscr{I}(X_{i,j})^2)}$, and $\|X\|_{1,2} := \sum_j \|X_{:,j}\|_2$. We use $\|X\|_{0,2}$ to denote the number of non-zero columns in X and use $\operatorname{vec}(X) \in \mathbb{C}^{d^2}$ to denote the vectorization of $X \in \mathbb{C}^{d\times d}$. Throughout this paper, we use $A \gtrsim B$ to denote $A \ge C_0 B$, where $C_0 \in \mathbb{R}_+$ is an absolute constant. The notation $\lesssim$ can be defined similarly.

E. Organization

The paper is organized as follows. After introducing some useful lemmas in Section 2, we present the proofs of Theorem I.2 and of Corollary I.4 in Section 3. The proof of Lemma I.1 is presented in Section 4. In Section 5, we introduce the DCA algorithm for solving the sparse PhaseLiftOff model, which can find a stationary point. We make a lot of numerical experiments, which show our method has better performance over the other known algorithms for sparse phase retrieval.

II. Preliminaries and Lemmas

Before the proof of the main theorems, we introduce some auxiliary lemmas as below.

Lemma II.1: [29] If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^d$, and $\langle \mathbf{x}, \mathbf{y}\rangle \ge 0$. Then

$$\|\mathbf{x}\mathbf{x}^* - \mathbf{y}\mathbf{y}^*\|_F^2 \ge \frac{1}{2}\|\mathbf{x}\|_2^2\|\mathbf{x} - \mathbf{y}\|_2^2.$$

The following lemma follows from the proof of Theorem 3.1 in [30]. We include a proof here for completeness.

Lemma II.2: Suppose that $\langle X, Y\rangle = 0$, where $X, Y \succeq 0$. If $\operatorname{rank}(X) = r \ge 1$, then

$$\left\|\lambda\left(\mathbf{I} - \frac{X}{\|X\|_F}\right) - Y\right\|_F \ge \lambda(\sqrt{r} - 1). \tag{13}$$

Here λ is a non-negative constant.

Proof: A simple calculation shows that

$$\begin{aligned}&\left\|\lambda\left(\mathbf{I} - \frac{X}{\|X\|_F}\right) - Y\right\|_F \\ &\ge \|\lambda\mathbf{I} - Y\|_F - \lambda\left\|\frac{X}{\|X\|_F}\right\|_F \\ &= \|\lambda\mathbf{I} - Y\|_F - \lambda,\end{aligned} \tag{14}$$

We next present a lower bound of $\|\lambda\mathbf{I} - Y\|_F$. Suppose that the singular value decomposition of X is in the form of

$$X = \sum_{i=1}^r \sigma_i \mathbf{u}_i\mathbf{u}_i^*,$$

where $U_1 = (\mathbf{u}_1, \dots, \mathbf{u}_r) \in \mathbb{C}^{d\times r}$, and $\sigma_i > 0$, for $i = 1, \dots, r$. Construct $U_2 \in \mathbb{C}^{d\times(d-r)}$, which satisfies $\mathbf{I} = U_1U_1^* + U_2U_2^*$ and $U_1^*U_2 = 0$. Then we have

$$\begin{aligned}&\|\lambda\mathbf{I} - Y\|_F^2 \\ &= \|\lambda(U_1U_1^* + U_2U_2^*) - Y\|_F^2 = \|\lambda U_1U_1^* + (\lambda U_2U_2^* - Y)\|_F^2\end{aligned}$$

$$= \|\lambda U_1U_1^*\|_F^2 + \|\lambda U_2U_2^* - Y\|_F^2 + 2\lambda\langle U_1U_1^*, \lambda U_2U_2^* - Y\rangle$$
$$= \|\lambda U_1U_1^*\|_F^2 + \|\lambda U_2U_2^* - Y\|_F^2 - 2\lambda\langle U_1U_1^*, Y\rangle. \quad (15)$$

The last line follows from $U_1^*U_2 = 0$. Since $\sigma_i > 0$ and $Y \succeq 0$, the condition $\langle X, Y\rangle = 0$ implies that

$$0 = \langle X, Y\rangle = \sum_{i=1}^r \sigma_i\langle \mathbf{u}_i\mathbf{u}_i^*, Y\rangle = \sum_{i=1}^r \sigma_i \mathrm{Tr}(\mathbf{u}_i^* Y \mathbf{u}_i),$$

which leads to $\langle \mathbf{u}_i\mathbf{u}_i^*, Y\rangle = 0, i = 1, \ldots, r$. Therefore, it obtain that

$$\langle U_1U_1^*, Y\rangle = \sum_{i=1}^r \langle \mathbf{u}_i\mathbf{u}_i^*, Y\rangle = 0, \quad (16)$$

and (15) becomes

$$\|\lambda\mathbf{I} - Y\|_F^2 = \|\lambda U_1U_1^*\|_F^2 + \|\lambda U_2U_2^* - Y\|_F^2 \geq \|\lambda U_1U_1^*\|_F^2 = \lambda^2 r. \quad (17)$$

Combining (14) and (17), we have

$$\left\|\lambda\left(\mathbf{I} - \frac{X}{\|X\|_F}\right) - Y\right\|_F \geq \|\lambda\mathbf{I} - Y\|_F - \lambda \geq \lambda(\sqrt{r} - 1).$$

■

III. Proofs of Theorem I.2 and of Corollary I.4

The aim of this section is to present the proofs of Theorem I.2 and of Corollary I.4.

Proof of Theorem I.2: Set

$$\mathbf{x}^{\#} := \operatorname*{arg\,min}_{\mathbf{x}\in\mathbb{C}^d} \mu\|\mathbf{x}\|_1^2 + \frac{1}{2}\sum_{i=1}^m (|\langle \mathbf{a}_i, \mathbf{x}\rangle|^2 - b_i)^2. \quad (18)$$

Noting $\exp(i\theta)\mathbf{x}^{\#}$ is also a solution to (18) for any $\theta \in \mathbb{R}$, without loss of generality, we can assume that

$$\langle \mathbf{x}^{\#}, \mathbf{x}_0\rangle \in \mathbb{R} \quad \text{and} \quad \langle \mathbf{x}^{\#}, \mathbf{x}_0\rangle \geq 0.$$

Then a simple observation is that $X^{\#}$ is the solution to (7) if and only if $X^{\#} = \mathbf{x}^{\#}(\mathbf{x}^{\#})^*$.

Set $X_0 := \mathbf{x}_0\mathbf{x}_0^*$ and

$$H := X^{\#} - X_0 = \mathbf{x}^{\#}(\mathbf{x}^{\#})^* - \mathbf{x}_0\mathbf{x}_0^*.$$

To prove the conclusion, it is enough to consider the upper bound of $\|H\|_F$. Set $T_0 := \mathrm{supp}(\mathbf{x}_0)$. Set T_1 as the index set which contains the indices of the ak largest elements of $\mathbf{x}^{\#}_{T_0^c}$ in magnitude, and T_2 contains the indices of the next ak largest elements, and so on. For simplicity, we set $T_{01} := T_0 \cup T_1$ and

$$\bar{H} := H_{T_{01},T_{01}}. \quad (19)$$

Note that

$$\|H\|_F \leq \|\bar{H}\|_F + \sum_{i\geq 2, j\geq 2} \|H_{T_i,T_j}\|_F + 2\sum_{j\geq 2, i=0,1} \|H_{T_i,T_j}\|_F.$$

So, it is enough to present upper bounds for

$$\|\bar{H}\|_F, \sum_{i\geq 2, j\geq 2} \|H_{T_i,T_j}\|_F, \text{ and } \sum_{j\geq 2, i=0,1} \|H_{T_i,T_j}\|_F.$$

Step 1: We first consider $\sum_{i\geq 2, j\geq 2} \|H_{T_i,T_j}\|_F$.

According to

$$\mu\|X^{\#}\|_1 + \frac{1}{2}\|\mathcal{A}(X^{\#}) - \mathbf{b}\|_2^2 \leq \mu\|X_0\|_1 + \frac{1}{2}\|\mathcal{A}(X_0) - \mathbf{b}\|_2^2,$$

we can obtain that

$$\mu\|H - H_{T_0,T_0}\|_1 \leq \mu\|H_{T_0,T_0}\|_1 + \|\mathbf{w}\|_2\|\mathcal{A}(H)\|_2 - \frac{1}{2}\|\mathcal{A}(H)\|_2^2. \quad (20)$$

Here, we use

$$\|X_0\|_1 = \|H_{T_0,T_0} - X^{\#}_{T_0,T_0}\|_1 \leq \|H_{T_0,T_0}\|_1 + \|X^{\#}_{T_0,T_0}\|_1,$$

and

$$\|X^{\#}\|_1 - \|X^{\#}_{T_0,T_0}\|_1 = \|X^{\#} - X^{\#}_{T_0,T_0}\|_1 = \|H - H_{T_0,T_0}\|_1.$$

Therefore, we have

$$\begin{aligned}
&\sum_{i\geq 2, j\geq 2} \|H_{T_i,T_j}\|_F \\
&= \sum_{i\geq 2, j\geq 2} \|\mathbf{x}^{\#}_{T_i}\|_2 \cdot \|\mathbf{x}^{\#}_{T_j}\|_2 = \left(\sum_{i\geq 2}\|\mathbf{x}^{\#}_{T_i}\|_2\right)^2 \\
&\leq \frac{1}{ak}\|\mathbf{x}^{\#}_{T_0^c}\|_1^2 = \frac{1}{ak}\|H_{T_0^c,T_0^c}\|_1 \leq \frac{1}{ak}\|H - H_{T_0,T_0}\|_1 \\
&\leq \frac{1}{ak}\|H_{T_0,T_0}\|_1 + \frac{\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu ak}\|\mathcal{A}(H)\|_2^2 \\
&\leq \frac{1}{a}\|H_{T_0,T_0}\|_F + \frac{\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu ak}\|\mathcal{A}(H)\|_2^2 \\
&\leq \frac{1}{a}\|\bar{H}\|_F + \frac{\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu ak}\|\mathcal{A}(H)\|_2^2. \quad (21)
\end{aligned}$$

The second line based on $\|\mathbf{x}^{\#}_{T_j}\|_2 \leq \|\mathbf{x}^{\#}_{T_{j-1}}\|_1/\sqrt{ak}$, and the third line follows from (20).

Step 2: We next consider $\sum_{j\geq 2, i=0,1} \|H_{T_i,T_j}\|_F$.

Using

$$\|\mathbf{x}^{\#}_{T_j}\|_2 \leq \|\mathbf{x}^{\#}_{T_{j-1}}\|_1/\sqrt{ak},$$

we obtain that

$$\begin{aligned}
&\sum_{j\geq 2} \|H_{T_i,T_j}\|_F \\
&= \|\mathbf{x}^{\#}_{T_i}\|_2 \cdot \sum_{j\geq 2}\|\mathbf{x}^{\#}_{T_j}\|_2 \leq \frac{1}{\sqrt{ak}}\|\mathbf{x}^{\#}_{T_0^c}\|_1\|\mathbf{x}^{\#}_{T_i}\|_2 \\
&\leq \frac{1}{\sqrt{a}}\|\mathbf{x}^{\#}_{T_{01}} - \mathbf{x}_0\|_2\|\mathbf{x}^{\#}_{T_i}\|_2 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu ak}}\|\mathbf{x}^{\#}_{T_i}\|_2 \\
&\leq \frac{\sqrt{2}}{\sqrt{a}}\|\mathbf{x}^{\#}_{T_{01}}(\mathbf{x}^{\#}_{T_{01}})^* - \mathbf{x}_0\mathbf{x}_0^*\|_F + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu ak}}\|\mathbf{x}^{\#}_{T_i}\|_2 \\
&= \frac{\sqrt{2}}{\sqrt{a}}\|\bar{H}\|_F + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu ak}}\|\mathbf{x}^{\#}_{T_i}\|_2, \quad (22)
\end{aligned}$$

where $i \in \{0,1\}$. Here, the third line follows from Lemma II.1 and the second line follows from

$$\|\mathbf{x}^{\#}_{T_0^c}\|_1 \le \sqrt{k}\|\mathbf{x}^{\#}_{T_0} - \mathbf{x}_0\|_2 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}}. \tag{23}$$

Indeed, noting that

$$\mu\|\mathbf{x}^{\#}(\mathbf{x}^{\#})^*\|_1 + \frac{1}{2}\|\mathcal{A}(\mathbf{x}^{\#}(\mathbf{x}^{\#})^*) - \mathbf{b}\|_2^2 \le \mu\|\mathbf{x}_0\mathbf{x}_0^*\|_1 + \frac{1}{2}\|\mathcal{A}(\mathbf{x}_0(\mathbf{x}_0)^*) - \mathbf{b}\|_2^2,$$

we obtain that

$$\mu\|\mathbf{x}^{\#}\|_1^2 + \frac{1}{2}\|\mathcal{A}(\mathbf{x}^{\#}(\mathbf{x}^{\#})^*) - \mathbf{b}\|_2^2 \le \mu\|\mathbf{x}_0\|_1^2 + \frac{1}{2}\|\mathcal{A}(\mathbf{x}_0(\mathbf{x}_0)^*) - \mathbf{b}\|_2^2$$

which implies

$$\begin{aligned}
&\|\mathbf{x}^{\#}\|_1\\
&\le \sqrt{\|\mathbf{x}_0\|_1^2 + \frac{1}{2\mu}\|\mathcal{A}(\mathbf{x}_0(\mathbf{x}_0)^*) - \mathbf{b}\|_2^2 - \frac{1}{2\mu}\|\mathcal{A}(\mathbf{x}^{\#}(\mathbf{x}^{\#})^*) - \mathbf{b}\|_2^2}\\
&\le \sqrt{\|\mathbf{x}_0\|_1^2 + \frac{1}{\mu}\|\mathcal{A}(H)\|_2\|\mathcal{A}(\mathbf{x}_0(\mathbf{x}_0)^*) - \mathbf{b}\|_2 - \frac{1}{2\mu}\|\mathcal{A}(H)\|_2^2}\\
&= \sqrt{\|\mathbf{x}_0\|_1^2 + \frac{\|\mathbf{w}\|_2}{\mu}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu}\|\mathcal{A}(H)\|_2^2}\\
&\le \|\mathbf{x}_0\|_1 + \sqrt{\max\left\{0, \frac{\|\mathbf{w}\|_2}{\mu}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu}\|\mathcal{A}(H)\|_2^2\right\}}\\
&\le \|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}}.
\end{aligned} \tag{24}$$

Therefore, we have

$$\begin{aligned}
\|\mathbf{x}^{\#}_{T_0^c}\|_1 &\le -\|\mathbf{x}^{\#}_{T_0}\|_1 + \|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}}\\
&\le \|\mathbf{x}^{\#}_{T_0} - \mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}}\\
&\le \sqrt{k}\|\mathbf{x}^{\#}_{T_0} - \mathbf{x}_0\|_2 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}},
\end{aligned}$$

which implies (23). Combing (21) and (22), we have

$$\begin{aligned}
&\sum_{i\ge 2, j\ge 2}\|H_{T_i,T_j}\|_F + 2\sum_{j\ge 2, i=0,1}\|H_{T_i,T_j}\|_F\\
&\le \left(\frac{1}{a} + \frac{4\sqrt{2}}{\sqrt{a}}\right)\|\bar{H}\|_F + \frac{2\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\|\mathbf{x}^{\#}_{T_{01}}\|_2\\
&\quad + \frac{\|\mathbf{w}\|_2}{\mu a k}\|\mathcal{A}(H)\|_2 - \frac{1}{2\mu a k}\|\mathcal{A}(H)\|_2^2\\
&\le \left(\frac{1}{a} + \frac{4\sqrt{2}}{\sqrt{a}}\right)\|\bar{H}\|_F + \frac{2\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\|\mathbf{x}^{\#}_{T_{01}}\|_2 + \frac{\|\mathbf{w}\|_2^2}{2\mu a k}.
\end{aligned} \tag{25}$$

Step 3: We claim that

$$\begin{aligned}
\|\bar{H}\|_F \le{}& \frac{1}{c - \frac{C}{a} - \frac{4\sqrt{2}C}{\sqrt{a}}} 2C\frac{\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\|\mathbf{x}^{\#}_{T_{01}}\|_2\\
&+ \frac{1}{c - \frac{C}{a} - \frac{4\sqrt{2}C}{\sqrt{a}}}\frac{\mu a k}{2\,C}\left(\frac{C\|\mathbf{w}\|_2}{\mu a k} + \frac{1}{\sqrt{m}}\right)^2.
\end{aligned} \tag{26}$$

Combining (25) and (26), we obtain that

$$\begin{aligned}
&\|H\|_F\\
&\le \|\bar{H}\|_F + \sum_{i\ge 2, j\ge 2}\|H_{T_i,T_j}\|_F + 2\sum_{j\ge 2, i=0,1}\|H_{T_i,T_j}\|_F\\
&\le \left(1 + \frac{1}{a} + \frac{4\sqrt{2}}{\sqrt{a}}\right)\|\bar{H}\|_F + \frac{2\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\|\mathbf{x}^{\#}_{T_{01}}\|_2 + \frac{\|\mathbf{w}\|_2^2}{2\mu a k}\\
&\le (2C\alpha + 2)\frac{\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\|\mathbf{x}^{\#}_{T_{01}}\|_2 + \frac{\|\mathbf{w}\|_2^2}{2\mu a k}\\
&\quad + \alpha\cdot\frac{\mu a k}{2\,C}\left(\frac{C\|\mathbf{w}\|_2}{\mu a k} + \frac{1}{\sqrt{m}}\right)^2\\
&\le (2C\alpha + 2)\frac{\|\mathbf{w}\|_2}{\sqrt{\mu a k}}\left(\|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}}\right) + \frac{\|\mathbf{w}\|_2^2}{2\mu a k}\\
&\quad + \alpha\cdot\frac{\mu a k}{2\,C}\left(\frac{C\|\mathbf{w}\|_2}{\mu a k} + \frac{1}{\sqrt{m}}\right)^2,
\end{aligned}$$

which leads to the conclusion. Here, the fourth line is based on

$$\|\mathbf{x}^{\#}_{T_{01}}\|_2 \le \|\mathbf{x}^{\#}_{T_{01}}\|_1 \le \|\mathbf{x}^{\#}\|_1 \le \|\mathbf{x}_0\|_1 + \frac{\|\mathbf{w}\|_2}{\sqrt{2\mu}},$$

where the last inequality follows from (24).

It remains to prove (26). Note that

$$\begin{aligned}
\frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2 &\ge \frac{1}{m}\|\mathcal{A}(H)\|_1\\
&\ge \frac{1}{m}\|\mathcal{A}(\bar{H})\|_1 - \frac{1}{m}\|\mathcal{A}(H - \bar{H})\|_1,
\end{aligned}$$

which implies

$$\frac{1}{m}\|\mathcal{A}(\bar{H})\|_1 \le \frac{1}{m}\|\mathcal{A}(H - \bar{H})\|_1 + \frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2. \tag{27}$$

Here we can see that

$$\begin{aligned}
&\frac{1}{m}\|\mathcal{A}(H - \bar{H})\|_1\\
&\le \sum_{i=0,1}\frac{1}{m}\|\mathcal{A}(H_{T_i,T_{01}^c} + H_{T_{01}^c,T_i})\|_1 + \frac{1}{m}\|\mathcal{A}(H_{T_{01}^c,T_{01}^c})\|_1.
\end{aligned}$$

For $i = 0,1$, since $\mathcal{A}(\cdot)$ satisfy the RIP condition of order $(2, 2ak)$ with upper RIP constant C, we have

$$\begin{aligned}
&\frac{1}{m}\|\mathcal{A}(H_{T_i,T_{01}^c} + H_{T_{01}^c,T_i})\|_1\\
&= \frac{1}{m}\left\|\sum_{j\ge 2}\mathcal{A}(H_{T_i,T_j} + H_{T_j,T_i})\right\|_1
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{m}\sum_{j\geq 2}\|\mathcal{A}(H_{T_i,T_j}+H_{T_j,T_i})\|_1\\
&= \frac{1}{m}\sum_{j\geq 2}\|\mathcal{A}(\mathbf{x}^{\#}_{T_i}(\mathbf{x}^{\#}_{T_j})^*+\mathbf{x}^{\#}_{T_j}(\mathbf{x}^{\#}_{T_i})^*)\|_1\\
&\leq 2C\sum_{j\geq 2}\|\mathbf{x}^{\#}_{T_i}\|_2\|\mathbf{x}^{\#}_{T_j}\|_2 \leq \frac{2\,C}{\sqrt{ak}}\|\mathbf{x}^{\#}_{T_0^c}\|_1\|\mathbf{x}^{\#}_{T_i}\|_2\\
&\leq \frac{2\,C}{\sqrt{a}}\|\bar{H}\|_F+\frac{2C\|\mathbf{w}\|_2}{\sqrt{2\mu ak}}\|\mathbf{x}^{\#}_{T_i}\|_2. \qquad (28)
\end{aligned}
$$

Here, the last line follows from (22). On the other hand, based on (21), we have

$$
\begin{aligned}
&\frac{1}{m}\|\mathcal{A}(H_{T_{01}^c,T_{01}^c})\|_1\\
&\leq \frac{1}{m}\left\|\sum_{i,j\geq 2,i\neq j}\mathcal{A}(H_{T_i,T_j}+H_{T_j,T_i})\right\|_1+\frac{1}{m}\left\|\sum_{i\geq 2}\mathcal{A}(H_{T_i,T_i})\right\|_1\\
&\leq \frac{C}{a}\|\bar{H}\|_F+\frac{C\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2-\frac{C}{2\mu ak}\|\mathcal{A}(H)\|_2^2. \qquad (29)
\end{aligned}
$$

As $\mathcal{A}(\cdot)$ satisfy the RIP condition of order $(2, 2ak)$ with lower RIP constant $c>0$, combining (27), (28) and (29), we obtain that

$$
\begin{aligned}
&c\|\bar{H}\|_F\\
&\leq \frac{1}{m}\|\mathcal{A}(\bar{H})\|_1 \leq \frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2+\frac{1}{m}\|\mathcal{A}(H-\bar{H})\|_1\\
&\leq \frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2+\sum_{i=0,1}\frac{1}{m}\|\mathcal{A}(H_{T_i,T_{01}^c}+H_{T_{01}^c,T_i})\|_1\\
&\quad+\frac{1}{m}\|\mathcal{A}(H_{T_{01}^c,T_{01}^c})\|_1\\
&\leq \frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2+2C\frac{\|\mathbf{w}\|_2}{\sqrt{\mu ak}}(\|\mathbf{x}^{\#}_{T_0}\|_2+\|\mathbf{x}^{\#}_{T_1}\|_2)\\
&\quad+\left(\frac{C}{a}+\frac{4\sqrt{2}C}{\sqrt{a}}\right)\|\bar{H}\|_F\\
&\quad+\frac{C\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2-\frac{C}{2\mu ak}\|\mathcal{A}(H)\|_2^2\\
&\leq \frac{1}{\sqrt{m}}\|\mathcal{A}(H)\|_2+2C\frac{\|\mathbf{w}\|_2}{\sqrt{\mu ak}}\|\mathbf{x}^{\#}_{T_{01}}\|_2\\
&\quad+\left(\frac{C}{a}+\frac{4\sqrt{2}C}{\sqrt{a}}\right)\|\bar{H}\|_F\\
&\quad+\frac{C\|\mathbf{w}\|_2}{\mu ak}\|\mathcal{A}(H)\|_2-\frac{C}{2\mu ak}\|\mathcal{A}(H)\|_2^2, \qquad (30)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left(c-\frac{C}{a}-\frac{4\sqrt{2}C}{\sqrt{a}}\right)\|\bar{H}\|_F\\
&\leq 2C\frac{\|\mathbf{w}\|_2}{\sqrt{\mu ak}}\|\mathbf{x}^{\#}_{T_{01}}\|_2+\left(\frac{C\|\mathbf{w}\|_2}{\mu ak}+\frac{1}{\sqrt{m}}\right)\|\mathcal{A}(H)\|_2\\
&\quad-\frac{C}{2\mu ak}\|\mathcal{A}(H)\|_2^2\\
&\leq 2C\frac{\|\mathbf{w}\|_2}{\sqrt{\mu ak}}\|\mathbf{x}^{\#}_{T_{01}}\|_2+\frac{\mu ak}{2\,C}\left(\frac{C\|\mathbf{w}\|_2}{\mu ak}+\frac{1}{\sqrt{m}}\right)^2.
\end{aligned}
$$

It leads to the inequality (26). ■

We next present the proof of Corollary I.4.

Proof of Corollary I.4: Theorem I.3 implies that the following holds with high probability provided $m \gtrsim k\log(d/k)$:

$$
0.12\|X\|_F \leq \frac{1}{m}\|\mathcal{A}(X)\|_1 \leq 2.45\|X\|_F,
$$

for all $X\in\mathbb{H}^{n\times n}$ with $\mathrm{rank(X)}\leq \mathrm{k}$ and $\|X\|_{0,2}\leq k$ (also $\|X^*\|_{0,2}\leq k$). Thus we can take $c=0.12$ and $C=2.45$ in (11). To guarantee (11) holds, it is enough to take a as a constant which is larger than $(8\cdot 2.45/0.12)^2$. Hence, using Theorem I.2, we obtain that

$$
\begin{aligned}
&\|\mathbf{x}^{\#}(\mathbf{x}^{\#})^*-\mathbf{x}_0\mathbf{x}_0^*\|_F\\
&\lesssim \frac{\|\mathbf{w}\|_2}{\sqrt{\mu k}}\left(\|\mathbf{x}_0\|_1+\frac{\|\mathbf{w}\|_2}{\sqrt{\mu}}\right)\\
&\quad+\frac{\|\mathbf{w}\|_2^2}{2\mu k}+\mu k\left(\frac{\|\mathbf{w}\|_2}{\mu k}+\frac{1}{\sqrt{m}}\right)^2.
\end{aligned}
$$

■

IV. Proof of Lemma I.1

Denote $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ as the set of symmetric real $d\times d$ matrices, and $\mathbb{R}^{d\times d}_{\mathrm{skew}}$ as the set of skew-symmetric real $d\times d$ matrices. If $X\in\mathbb{H}^{d\times d}$, then X can be written as $X=X_1+iX_2$, where $X_1\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $X_2\in\mathbb{R}^{d\times d}_{\mathrm{skew}}$ are the real and imaginary parts of X. Thus the set $\{X\in\mathbb{H}^{d\times d}\ :\ X\succeq 0\}$ corresponds to

$$
\begin{aligned}
\mathbb{H}^{d\times d}_{+} := \Bigg\{ \begin{bmatrix} X_1\\ X_2\end{bmatrix} &: (X_1,\ X_2)\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^{d\times d}_{\mathrm{skew}},\\
&\mathbf{z}_1^{\mathrm{T}}X_1\mathbf{z}_1+\mathbf{z}_2^{\mathrm{T}}X_1\mathbf{z}_2+\mathbf{z}_2^{\mathrm{T}}X_2\mathbf{z}_1-\mathbf{z}_1^{\mathrm{T}}X_2\mathbf{z}_2\geq 0,\\
&\text{for all } \mathbf{z}_1,\mathbf{z}_2\in\mathbb{R}^d\Bigg\}.
\end{aligned}
$$

Let $\widetilde{\mathcal{A}}:\mathbb{R}^{2d\times d}\to\mathbb{R}^m$ be defined by

$$
\begin{aligned}
\begin{bmatrix} X_1\\ X_2\end{bmatrix} &\mapsto (\mathscr{R}(\mathbf{a}_i)^{\mathrm{T}}X_1\mathscr{R}(\mathbf{a}_i)+\mathscr{I}(\mathbf{a}_i)^{\mathrm{T}}X_1\mathscr{I}(\mathbf{a}_i)\\
&\quad+\mathscr{I}(\mathbf{a}_i)^{\mathrm{T}}X_2\mathscr{R}(\mathbf{a}_i)-\mathscr{R}(\mathbf{a}_i)^{\mathrm{T}}X_2\mathscr{I}(\mathbf{a}_i))_{i=1}^m. \qquad (31)
\end{aligned}
$$

Then $\mathcal{A}(X) = \widetilde{A}(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix})$. By a simple calculation, its conjugate map $\widetilde{\mathcal{A}}^* : \mathbb{R}^m \to \mathbb{R}^{2d\times d}$ is given by

$$(b_i)_{i=1}^m \mapsto \begin{bmatrix} \sum_{i=1}^m b_i \left(\mathscr{R}(\mathbf{a}_i)\mathscr{R}(\mathbf{a}_i)^{\mathrm{T}} + \mathscr{I}(\mathbf{a}_i)\mathscr{I}(\mathbf{a}_i)^{\mathrm{T}}\right) \\ \sum_{i=1}^m b_i \left(\mathscr{I}(\mathbf{a}_i)\mathscr{R}(\mathbf{a}_i)^{\mathrm{T}} - \mathscr{R}(\mathbf{a}_i)\mathscr{I}(\mathbf{a}_i)^{\mathrm{T}}\right) \end{bmatrix}. \tag{32}$$

For $X = X_1 + iX_2 \in \mathbb{C}^{d\times d}$, $\|X\|_1$ and $\|X\|_F$ can also be written as

$$\|X\|_1 = \left\| \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} \right\|_{1,2} = \sum_{i,j} \sqrt{[X_1]_{i,j}^2 + [X_2]_{i,j}^2},$$

and

$$\|X\|_F = \left\| \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right\|_F.$$

Using the notations above, we recast the model (8) as follows.

$$\min_{X_1,X_2} \lambda \left(\mathrm{Tr}(X_1) - \left\| \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right\|_F \right) + \mu \left\| \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} \right\|_{1,2}$$
$$+ \frac{1}{2} \left\| \widetilde{\mathcal{A}}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right) - \mathbf{b} \right\|_2^2 \quad \text{s.t.} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{H}_+^{d\times d}. \tag{33}$$

If $(X_1^\#, X_2^\#)$ is a minimizer of (33), then the optimal solution $X^\#$ of (8) satisfies $X^\# = X_1^\# + iX_2^\#$.

In order to prove Lemma I.1, we first introduce some technical lemmas in convex optimization and matrix theory. Assume that $\Omega \subset \mathbb{R}^n$. We use $T_\Omega(\mathbf{x}^\#)$ and $T_\Omega(\mathbf{x}^\#)^*$ to denote the tangent cone of Ω at $\mathbf{x}^\# \in \Omega$ and its dual cone, respectively. Particularly, we have

Proposition IV.1: If Ω is a convex cone in $\mathbb{R}^n$ and $\mathbf{x}^\# \in \Omega$, then

$$T_\Omega(\mathbf{x}^\#)^* = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y}, \mathbf{x}\rangle \le 0 \text{ for all } \mathbf{x} \in \Omega, \text{ and } \langle \mathbf{y}, \mathbf{x}^\#\rangle = 0\}.$$

Proof: According to Proposition 4.6.3 in [2], we have

$$T_\Omega(\mathbf{x}^\#)^* = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y}, \mathbf{x} - \mathbf{x}^\#\rangle \le 0 \text{ for all } \mathbf{x} \in \Omega\}. \tag{34}$$

Assume that $\mathbf{y} \in T_\Omega(\mathbf{x}^\#)^*$. Then $\langle \mathbf{y}, \mathbf{x} - \mathbf{x}^\#\rangle \le 0$ for all $\mathbf{x} \in \Omega$. Since Ω is a cone, we have $\mathbf{x}^\#/2, 2\mathbf{x}^\# \in \Omega$. Taking $\mathbf{x} = 2\mathbf{x}^\#$, we obtain $\langle \mathbf{y}, \mathbf{x}^\#\rangle \le 0$. Similarly, taking $\mathbf{x} = \mathbf{x}^\#/2$, we have $\langle \mathbf{y}, \mathbf{x}^\#\rangle \ge 0$. We arrive at $\langle \mathbf{y}, \mathbf{x}^\#\rangle = 0$, which leads to $\langle \mathbf{y}, \mathbf{x}\rangle \le 0$ for all $\mathbf{x} \in \Omega$. ■

The following theorem provides some properties of local minimum on constrained model.

Proposition IV.2: [2, Proposition 4.7.3] Let $\mathbf{x}^\#$ be a local minimizer of the model:

$$\min_{\mathbf{x}\in\Omega} f_1(\mathbf{x}) + f_2(\mathbf{x}),$$

where f_1 is convex and f_2 is smooth over a subset Ω of $\mathbb{R}^n$. Assume that the tangent cone $T_\Omega(\mathbf{x}^\#)$ is convex. Then

$$-\nabla f_2(\mathbf{x}^\#) \in \partial f_1(\mathbf{x}^\#) + T_\Omega(\mathbf{x}^\#)^*.$$

We next present the sub-gradient set of $\partial(\|X\|_1)$:

Proposition IV.3: ([1]) Assume that $X = X_1 + iX_2$ with $X_1, X_2 \in \mathbb{R}^{d\times d}$. Then the subgradient set of $\|X\|_1$ in real space is

$$\partial \left(\left\| \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} \right\|_{1,2} \right)$$
$$:= \left\{ \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, G_1, G_2 \in \mathbb{R}^{d\times d} : \begin{array}{l} [G_1]_{i_1,i_2}^2 + [G_2]_{i_1,i_2}^2 \le 1, \text{ if } [X_1]_{i_1,i_2} = [X_2]_{i_1,i_2} = 0; \\ ([G_1]_{i_1,i_2}, [G_2]_{i_1,i_2}) = \dfrac{([X_1]_{i_1,i_2}, [X_2]_{i_1,i_2})}{\sqrt{[X_1]_{i_1,i_2}^2 + [X_2]_{i_1,i_2}^2}}, \text{ otherwise} \end{array} \right\} \tag{35}$$

Combining Proposition IV.1, Proposition IV.2 with Proposition IV.3, we have

Lemma IV.4: Assume that $(X_1^\#, X_2^\#)$ is a local minimizer of model (33). Then there exist $\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \in \mathbb{H}_+^{d\times d}$ and $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \in \partial(\| \begin{bmatrix} \mathrm{vec}(X_1^\#) \\ \mathrm{vec}(X_2^\#) \end{bmatrix} \|_{1,2})$ such that the followings hold:

1) Stationary condition:

$$\lambda \left(\begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix} \Big/ \left\| \begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix} \right\|_F \right) + \mu \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$
$$+ \widetilde{\mathcal{A}}^* \left(\widetilde{\mathcal{A}} \left(\begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix} \right) - \mathbf{b} \right) - \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} = 0; \tag{36}$$

2) Complementary slackness condition:

$$\left\langle \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}, \begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix} \right\rangle = 0. \tag{37}$$

Proof: Set

$$f_1(X_1, X_2) := \mu \left\| \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} \right\|_{1,2},$$

and

$$f_2(X_1, X_2) := \lambda \left(\mathrm{Tr}(X_1) - \left\| \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right\|_F \right) + \frac{1}{2} \left\| \widetilde{\mathcal{A}}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right) - \mathbf{b} \right\|_2^2,$$

and $\Omega := \mathbb{H}_+^{d\times d}$. Then f_1 is convex and f_2 is smooth. Since Ω is convex, we obtain that $T_\Omega(\begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix})$ is convex by Proposition 4.6.2 in [2]. According to Proposition IV.2, there exists $-\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \in T_\Omega(\begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix})^*$ such that the stationary condition (36) holds. Furthermore, we can use Proposition IV.1 to obtain the complementary slackness condition (37).

We remain to prove $\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \in \mathbb{H}_+^{d\times d}$. Take $T_1 = \mathbf{t}_1\mathbf{t}_1^{\mathrm{T}} + \mathbf{t}_2\mathbf{t}_2^{\mathrm{T}}$ and $T_2 = \mathbf{t}_2\mathbf{t}_1^{\mathrm{T}} - \mathbf{t}_1\mathbf{t}_2^{\mathrm{T}}$ for any fixed $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^d$. Then $\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \Omega$. By the definition of $T_\Omega(\begin{bmatrix} X_1^\# \\ X_2^\# \end{bmatrix})^*$ and Proposition IV.2, we obtain that

$$\left\langle -\begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}, \begin{bmatrix}T_1\\ T_2\end{bmatrix}\right\rangle \le 0,$$

which implies

$$\mathbf{t}_1^{\mathrm{T}}\Lambda_1\mathbf{t}_1+\mathbf{t}_2^{\mathrm{T}}\Lambda_1\mathbf{t}_2+\mathbf{t}_2^{\mathrm{T}}\Lambda_2\mathbf{t}_1-\mathbf{t}_1^{\mathrm{T}}\Lambda_2\mathbf{t}_2 \ge 0 \text{ for any } \mathbf{t}_1,\mathbf{t}_2\in\mathbb{R}^d. \quad (38)$$

If $(\Lambda_1,\Lambda_2)\in\mathbb{R}^{d\times d}_{\text{sym}}\times\mathbb{R}^{d\times d}_{\text{skew}}$, then we arrive at the conclusion. Otherwise, we can replace Λ_1 and Λ_2 by

$$\widetilde{\Lambda}_1 := \frac{\Lambda_1+\Lambda_1^{\mathrm{T}}}{2} \text{ and } \widetilde{\Lambda}_2 := \frac{\Lambda_2-\Lambda_2^{\mathrm{T}}}{2}.$$

Noting that $(\widetilde{\Lambda}_1,\widetilde{\Lambda}_2)\in\mathbb{R}^{d\times d}_{\text{sym}}\times\mathbb{R}^{d\times d}_{\text{skew}}$ and

$$\left\langle -\begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}, \begin{bmatrix}T_1\\ T_2\end{bmatrix}\right\rangle = \left\langle -\begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}, \begin{bmatrix}T_1\\ T_2\end{bmatrix}\right\rangle \le 0,$$

we obtain that $\begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}\in\mathbb{H}^{d\times d}_{+}$. After a simple calculation, we also have

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\Big/\left\|\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\|_F\right) + \mu\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix}$$
$$+\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right) - \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix} = 0,$$

and

$$\left\langle \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}, \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}, \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\rangle = 0,$$

where $\widetilde{G}_1 := \frac{G_1+G_1^{\mathrm{T}}}{2}, \widetilde{G}_2 := \frac{G_2-G_2^{\mathrm{T}}}{2}$ and

$$\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix} \in \partial\left(\left\|\begin{bmatrix}\mathrm{vec}(X_1^{\#})\\ \mathrm{vec}(X_2^{\#})\end{bmatrix}\right\|_{1,2}\right).$$

Therefore, the stationary condition (36) and complementary slackness condition (37) also hold for $\begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix} := \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}$. ■

We next present the proof of Lemma I.1.

Proof of Lemma I.1: Since $\frac{1}{2}\|\mathbf{b}\|_2^2 > \mu\|\mathbf{x}_0\|_1^2 + \frac{1}{2}\|\mathbf{w}\|_2^2$, we obtain that $X^{\#}\neq 0$.

We next consider the equivalent model (33) with global minimizer $(X_1^{\#}, X_2^{\#})$. According to Lemma IV.4, there exist $\begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}\in\mathbb{H}^{d\times d}_{+}$ and $\begin{bmatrix}G_1\\ G_2\end{bmatrix}\in\partial(\| \begin{bmatrix}\mathrm{vec}(X_1^{\#})\\ \mathrm{vec}(X_2^{\#})\end{bmatrix} \|_{1,2})$ such that the following holds:

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\Big/\left\|\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\|_F\right) + \mu\begin{bmatrix}G_1\\ G_2\end{bmatrix}$$
$$+\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right) - \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix} = 0; \quad (39)$$

and

$$\left\langle \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}, \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\rangle = 0. \quad (40)$$

According to (39), we obtain that

$$\begin{aligned}
&\left\|\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right)+\mu\begin{bmatrix}G_1\\ G_2\end{bmatrix}\right\|_F\\
&= \left\|\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\Big/\left\|\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right\|_F\right) - \begin{bmatrix}\Lambda_1\\ \Lambda_2\end{bmatrix}\right\|_F \qquad (41)\\
&= \left\|\lambda\left(\mathbf{I}-\frac{X^{\#}}{\|X^{\#}\|_F}\right)-\Lambda\right\|_F\\
&\ge \lambda(\sqrt{r}-1),
\end{aligned}$$

where $\Lambda := \Lambda_1 + i\Lambda_2 \in\mathbb{C}^{d\times d}$ and $r := \mathrm{rank}(X^{\#})$. The last inequality in (41) follows from (40) and Lemma II.2.

On the other hand, we have

$$\begin{aligned}
&\left\|\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right)+\mu\begin{bmatrix}G_1\\ G_2\end{bmatrix}\right\|_F\\
&\le \left\|\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right)\right\|_F + \mu\left\|\begin{bmatrix}G_1\\ G_2\end{bmatrix}\right\|_F\\
&\le \|\mathcal{A}\|\left\|\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{\#}\\ X_2^{\#}\end{bmatrix}\right)-\mathbf{b}\right\|_2 + \mu d \qquad (42)\\
&= \|\mathcal{A}\|\|\mathcal{A}(X^{\#})-\mathbf{b}\|_2 + \mu d\\
&\le \|\mathcal{A}\|\sqrt{2\mu\|\mathbf{x}_0\|_1^2+\|\mathbf{w}\|_2^2}+\mu d\\
&\le \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1+\|\mathbf{w}\|_2)+\mu d.
\end{aligned}$$

Here, the second inequality follows from Proposition IV.3 and $[G_1]^2_{i_1,i_2} + [G_1]^2_{i_1,i_2} \le 1$ for any $i_1, i_2 \in\{1,\ldots,d\}$. Combing (41) and (42), we obtain that

$$\lambda(\sqrt{r}-1) \le \mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1+\|\mathbf{w}\|_2). \quad (43)$$

Combing (9) and (43), we have

$$\begin{aligned}
&\frac{\sqrt{r}-1}{\sqrt{2}-1}\left(\mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1+\|\mathbf{w}\|_2)\right)\\
&< \mu d + \|\mathcal{A}\|(\sqrt{2\mu}\|\mathbf{x}_0\|_1+\|\mathbf{w}\|_2),
\end{aligned}$$

which implies $r = 1$. ■

V. Algorithms for Solving Sparse PhaseLiftOff

A. Introduction of the DCA

In this section, we establish an algorithm to solve the Sparse PhaseLiftOff model (8), which is stated in Algorithm 1.

Our algorithm is based on DCA, which is a descent method introduced by Tao and An [21], [22]. DCA is also studied in compressed sensing, and in matrix recovery problem (see [28], [30], [31]).

The step 6 of Algorithm 1 is to solve a subproblem (44). We suggest employing ADMM method [3] to solve it, which is shown in Algorithm 2. The convergence rate of ADMM was

Algorithm 1: The DCA for Solving Model (8).

1: **Input:** the map $\mathcal{A}$, the vector $\mathbf{b}$, the tolerance error tol ≥ 0, the parameters λ, μ and MAXiter.
2: **Output:** A matrix $X^{\#}$.
3: **Initial:** $X^0 = \mathbf{0}$.
4: **Loop: for** $k = 0$ **to** MAXiter

$$Y^k = \begin{cases} \frac{X^k}{\|X^k\|_F} & \text{if } X^k \neq \mathbf{0} \\ \mathbf{0} & \text{if } X^k = \mathbf{0} \end{cases}$$

$$X^{k+1} = \underset{X \succeq 0}{\operatorname{argmin}} \left\{ \frac{1}{2}\|\mathcal{A}(X) - \mathbf{b}\|_2^2 + \lambda \mathrm{Tr}(X) - \lambda\langle X, Y^k\rangle + \mu\|X\|_1 \right\} \quad (44)$$

If $\frac{\|X^k - X^{k-1}\|_F}{\max\{\|X^k\|_F, 1\}} \leq$ tol **then break**
5: $X^{\#} = X^k$.

established in [11]. To derive ADMM, we rewrite (44) as

$$\min_{X_3 \succeq 0} \frac{1}{2}\|\mathcal{A}(X_1) - \mathbf{b}\|_2^2 + \lambda\mathrm{Tr}(X_1) - \lambda\langle X_1, Y^k\rangle + \mu\|X_2\|_1$$
$$\text{s.t. } X_3 = X_1, X_3 = X_2 \quad (45)$$

The problem (45) is called global consensus problem [3, Equation (7.2)] with local variables X_1 and X_2 and a common global variable X_3. The augmented Lagrangian function corresponding to (45) is

$$\begin{aligned} &\mathcal{L}_\delta(X_1, X_2, X_3, Y_1, Y_2) \\ &= \frac{1}{2}\|\mathcal{A}(X_1) - \mathbf{b}\|_2^2 + \langle X_1, \lambda(\mathbf{I} - Y^k)\rangle + \mu\|X_2\|_1 + g_{\succeq}(X_3) \\ &\quad + \langle Y_1, X_1 - X_3\rangle + \langle Y_2, X_2 - X_3\rangle + \frac{\delta}{2}\|X_1 - X_3\|_F^2 \\ &\quad + \frac{\delta}{2}\|X_2 - X_3\|_F^2, \end{aligned}$$

where Y_1, Y_2 are dual variables, δ is augmented Lagrangian parameter and

$$g_{\succeq 0}(Z) = \begin{cases} 0 & \text{if } Z \succeq 0, \\ \infty & \text{otherwise.} \end{cases}$$

We can employ the standard ADMM to solve

$$\min_{X_1, X_2, X_3, Y_1, Y_2} \mathcal{L}_\delta(X_1, X_2, X_3, Y_1, Y_2), \quad (46)$$

which consists of updating on both the primal and dual variables [3, Equation (7.3)-Equation (7.5)]:

$$\begin{cases} X_1^{l+1} = \arg\min_{X_1} \mathcal{L}_\delta(X_1, X_2^l, X_3^l, Y_1^l, Y_2^l) \\ X_2^{l+1} = \arg\min_{X_2} \mathcal{L}_\delta(X_1^{l+1}, X_2, X_3^l, Y_1^l, Y_2^l) \\ X_3^{l+1} = \arg\min_{X_3} \mathcal{L}_\delta(X_1^{l+1}, X_2^{l+1}, X_3, Y_1^l, Y_2^l) \\ Y_1^{l+1} = Y_1^l + \delta(X_1^{l+1} - X_3^{l+1}) \\ Y_2^{l+1} = Y_2^l + \delta(X_2^{l+1} - X_3^{l+1}) \end{cases} \quad (47)$$

According to [3], δ can be fixed or adaptively updated following the rules below:

$$\delta^{l+1} = \begin{cases} 2\delta^l & \text{if } \|R^l\|_F > 10\|S^l\|_F \\ \delta^l/2 & \text{if } \|R^l\|_F < \frac{1}{10}\|S^l\|_F, \\ \delta^l & \text{otherwise} \end{cases}$$

where $\|R^l\|_F^2 = \|X_1^l - X_3^l\|_F^2 + \|X_2^l - X_3^l\|_F^2$, and $\|S^l\|_F^2 = 2(\delta^l)^2\|X_3^l - X_3^{l-1}\|_F^2$.

More explicitly, we state ADMM algorithm for solving (47) in Algorithm 2. In Algorithm 2, we use $\mathcal{S}_\lambda : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ to denote the soft-thresholding operator on each elements of the matrix, i.e.,

$$[\mathcal{S}_\lambda(Z)]_{i,j} = \begin{cases} (|Z_{i,j}| - \lambda)\frac{Z_{i,j}}{|Z_{i,j}|} & |Z_{i,j}| \geq \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We use $\mathcal{P}_{\succeq} : \mathbb{H}^{n\times n} \to \mathbb{H}^{n\times n}$ to denote the projection on the the positive semidefinite cone, i.e.,

$$\mathcal{P}_{\succeq}(X) = U\max\{\Sigma, \mathbf{0}\}U^*,$$

where $X = U\Sigma U^*$ is the eigenvalue decomposition of X.

We next analyze the computational complexity of our algorithm. It is mainly dominated by inverse operator $(\mathcal{A}^*\mathcal{A} + \delta\mathbf{I})^{-1}(\cdot)$ and eigenvalue decomposition in $\mathcal{P}_{\succeq}(\cdot)$ in Algorithm 2. We denote $\mathrm{diag}(X)$ as diagonals of matrix X and $\mathrm{diag}(\mathbf{x})$ as diagonal matrix generated from vector $\mathbf{x}$. Then $\mathcal{A}(X) = \mathrm{diag}(A^*XA)$ and $\mathcal{A}^*(\mathbf{x}) = A\mathrm{diag}(\mathbf{x})A^*$, where $A = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$. Noting that

$$\begin{aligned} &(\mathcal{A}^*\mathcal{A} + \delta\mathbf{I})^{-1}(X) \\ &= \frac{1}{\delta}(X - A\mathrm{diag}((A^*A \circ \overline{A^*A} + \delta\mathbf{I})^{-1}\mathrm{diag}(A^*XA))A^*) \end{aligned}$$

(see [30, Page 13]), we obtain that the time complexity for computing $(\mathcal{A}^*\mathcal{A} + \delta\mathbf{I})^{-1}(\cdot)$ is $\mathcal{O}(m^2d + m^3 + d^2m)$. Based on symmetric QR algorithm [14, Algorithm 8.3.3], the time complexity for computing $\mathcal{P}_{\succeq}(\cdot)$ is $\mathcal{O}(d^3)$. Therefore, the time complexity of DCA is $\mathcal{O}(l_{\mathrm{ADMM}}k_{\mathrm{DCA}}(d^3 + m^2d + m^3 + d^2m))$, where k_{DCA} and l_{ADMM} denote the number of iterations in Algorithm 1 and Algorithm 2, respectively.

The lifting method becomes computationally prohibitive as d or m increases. We would like to mention that one already develops many efficient technologies for computing an approximation of the inverse operator (see [12], [16]) and for large scale eigenvalue calculations (see [20]). Using these technologies to solve sparse PhaseLiftoff efficiently is an interesting future research problem.

B. The Convergence Property of Algorithm 1

The aim of this subsection is to study the convergence property of Algorithm 1. Motivated by the techniques developed in [30] and [31], we will show that Algorithm 1 converges to a stationary point. However, compared to models in [31] and [30], model (8) adds semidefinite constant of X and extra nonsmooth term $\|X\|_1$, which leads to the proof of convergence not simply covered by the analysis in [30] and [31].

Algorithm 2: ADMM for Solving the Subproblem (44).

1: **Input:** the map $\mathcal{A}$, the vector $\mathbf{b}$, k, $W=\lambda(\mathbf{I}-Y^k)$, the parameters λ,μ,δ and MAXiter.
2: **Output:** A matrix X^{k+1}.
3: **Initial:** $X_1^0=X_2^0=X_3^0=Y_1^0=Y_2^0=\mathbf{0}$.
4: **Loop: for** $l=0$ **to** MAXiter
 $X_1^{l+1}=(\mathcal{A}^*\mathcal{A}+\delta\mathbf{I})^{-1}(\mathcal{A}^*(\mathbf{b})-W+\delta X_3^l-Y_1^l)$
 $X_2^{l+1}=\mathcal{S}_{\mu/\delta}(X_3^l-\frac{1}{\delta}Y_2^l)$
 $X_3^{l+1}=\mathcal{P}_{\succeq}(\frac{1}{2}(X_1^{l+1}+X_2^{l+1})+\frac{1}{2\delta}(Y_1^l+Y_2^l))$
 $Y_1^{l+1}=Y_1^l+\delta(X_1^{l+1}-X_3^{l+1})$
 $Y_2^{l+1}=Y_2^l+\delta(X_2^{l+1}-X_3^{l+1})$
5: $X^{k+1}=X_3^l$.

For convenience, we set

$$F(X):=\lambda(\mathrm{Tr}(X)-\|X\|_F)+\mu\|X\|_1+\frac{1}{2}\|\mathcal{A}(X)-\mathbf{b}\|_2^2.$$

We first show that $\{F(X^k)\}_{k\geq 1}$ generated by Algorithm 1 is a monotonically decreasing sequence.

Lemma V.1: If $\{X^k\}_{k\geq 1}$ is a sequence generated by Algorithm 1, then we have

$$F(X^k)-F(X^{k+1})\geq 0,\quad \text{for all } k\geq 0.$$

Proof: We consider the kth iteration of Algorithm 1. Recall that X^{k+1} is the solution to (44) in Algorithm 1. Set $X^{k+1}:=X_1^{k+1}+iX_2^{k+1}$ and $Y^k:=Y_1^k+iY_2^k$ where $X_1^{k+1},X_2^{k+1},Y_1^k,Y_2^k\in\mathbb{R}^{d\times d}$. Take

$$f_1(X_1,X_2):=\mu\left\|\begin{bmatrix}\mathrm{vec}(X_1)\\ \mathrm{vec}(X_2)\end{bmatrix}\right\|_{1,2},$$

and

$$f_2(X_1,X_2) := \lambda\left(\mathrm{Tr}(X_1)-\left\langle\begin{bmatrix}Y_1^k\\ Y_2^k\end{bmatrix},\begin{bmatrix}X_1\\ X_2\end{bmatrix}\right\rangle\right)+\frac{1}{2}\left\|\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1\\ X_2\end{bmatrix}\right)-\mathbf{b}\right\|_2^2,$$

and $\Omega:=\mathbb{H}_+^{d\times d}$. Then f_1 is convex, f_2 is smooth, and $T_\Omega(\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix})$ is convex. According to Proposition IV.2, we have

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix}-\begin{bmatrix}Y_1^k\\ Y_2^k\end{bmatrix}\right)+\mu\begin{bmatrix}G_1^{k+1}\\ G_2^{k+1}\end{bmatrix}+\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix}\right)-\mathbf{b}\right) = \begin{bmatrix}\Lambda_1^{k+1}\\ \Lambda_2^{k+1}\end{bmatrix}, \tag{48}$$

and

$$\left\langle\begin{bmatrix}\Lambda_1^{k+1}\\ \Lambda_2^{k+1}\end{bmatrix},\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix}\right\rangle=0, \tag{49}$$

for some $\Lambda^{k+1}=\Lambda_1^{k+1}+i\Lambda_2^{k+1}$ with $\begin{bmatrix}\Lambda_1^{k+1}\\ \Lambda_2^{k+1}\end{bmatrix}\in\mathbb{H}_+^{d\times d}$, and $G^{k+1}=G_1^{k+1}+iG_2^{k+1}$ with $\begin{bmatrix}G_1^{k+1}\\ G_2^{k+1}\end{bmatrix}\in\partial(\|\begin{bmatrix}\mathrm{vec}(X_1^{k+1})\\ \mathrm{vec}(X_2^{k+1})\end{bmatrix}\|_{1,2})$. According to Proposition IV.3, if $[X_1^{k+1}]_{i_1,i_2}=[X_2^{k+1}]_{i_1,i_2}=0$, then $[G_1^{k+1}]_{i_1,i_2}^2+[G_2^{k+1}]_{i_1,i_2}^2\leq 1$; otherwise $([G_1^{k+1}]_{i_1,i_2},[G_2^{k+1}]_{i_1,i_2})=\frac{([X_1^{k+1}]_{i_1,i_2},[X_2^{k+1}]_{i_1,i_2})}{\sqrt{[X_1^{k+1}]_{i_1,i_2}^2+[X_2^{k+1}]_{i_1,i_2}^2}}$.

Using a similar method for proving Lemma IV.4, we obtain (49).

According to (48), we have

$$\left\langle\begin{bmatrix}X_1^k-X_1^{k+1}\\ X_2^k-X_2^{k+1}\end{bmatrix},\lambda\begin{bmatrix}\mathbf{I}-\mathbf{Y_1^k}\\ \mathbf{0}-\mathbf{Y_2^k}\end{bmatrix}+\mu\begin{bmatrix}G_1^{k+1}\\ G_2^{k+1}\end{bmatrix}+\widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix}\right)-\mathbf{b}\right)\right\rangle = \left\langle\begin{bmatrix}X_1^k-X_1^{k+1}\\ X_2^k-X_2^{k+1}\end{bmatrix},\begin{bmatrix}\Lambda_1^{k+1}\\ \Lambda_2^{k+1}\end{bmatrix}\right\rangle. \tag{50}$$

Combining (50),

$$\left\langle\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix},\begin{bmatrix}G_1^{k+1}\\ G_2^{k+1}\end{bmatrix}\right\rangle=\|X^{k+1}\|_1,$$

and

$$\left\langle\begin{bmatrix}X_1^k\\ X_2^k\end{bmatrix},\begin{bmatrix}Y_1^k\\ Y_2^k\end{bmatrix}\right\rangle=\|X^k\|_F,\quad \left\langle\begin{bmatrix}\Lambda_1^{k+1}\\ \Lambda_2^{k+1}\end{bmatrix},\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix}\right\rangle=0,$$

we obtain that

$$\begin{aligned}&\langle X^k,\Lambda^{k+1}\rangle\\ &=\lambda\mathrm{Tr}(X^k-X^{k+1})-\lambda\|X^k\|_F+\lambda\langle X^{k+1},Y^k\rangle\\ &\quad+\mu\langle X^k,G^{k+1}\rangle-\mu\|X^{k+1}\|_1\\ &\quad+\langle\mathcal{A}(X^k-X^{k+1}),\mathcal{A}(X^{k+1})-\mathbf{b}\rangle,\end{aligned} \tag{51}$$

since $\widetilde{\mathcal{A}}(\begin{bmatrix}X_1^{k+1}\\ X_2^{k+1}\end{bmatrix})=\mathcal{A}(X^{k+1})$ and $\widetilde{\mathcal{A}}(\begin{bmatrix}X_1^k-X_1^{k+1}\\ X_2^k-X_2^{k+1}\end{bmatrix})=\mathcal{A}(X^k-X^{k+1})$ with $X^k=X_1^k+iX_2^k$ and $X^{k+1}=X_1^{k+1}+iX_2^{k+1}$. Combining

$$\begin{aligned}&F(X^k)-F(X^{k+1})\\ &=\frac{1}{2}\|\mathcal{A}(X^{k+1}-X^k)\|_2^2+\langle\mathcal{A}(X^k-X^{k+1}),\mathcal{A}(X^{k+1})-\mathbf{b}\rangle\\ &\quad+\lambda(\mathrm{Tr}(X^k-X^{k+1})-\|X^k\|_F+\|X^{k+1}\|_F)\\ &\quad+\mu(\|X^k\|_1-\|X^{k+1}\|_1),\end{aligned}$$

and (51), we arrive at

$$\begin{aligned}F(X^k)-F(X^{k+1})&=\frac{1}{2}\|\mathcal{A}(X^{k+1}-X^k)\|_2^2+\langle X^k,\Lambda^{k+1}\rangle\\ &\quad+\mu(\|X^k\|_1-\langle X^k,G^{k+1}\rangle)\\ &\quad+\lambda(\|X^{k+1}\|_F-\langle X^{k+1},Y^k\rangle)\\ &\geq 0.\end{aligned} \tag{52}$$

Here, the last inequality follows from $\|X^k\|_1-\langle X^k,G^{k+1}\rangle\geq 0$, $\|X^{k+1}\|_F-\langle X^{k+1},Y^k\rangle\geq 0$, and $\langle X^k,\Lambda^{k+1}\rangle\geq 0$ since $\|G^{k+1}\|_\infty\leq 1$, $\|Y^k\|_F\leq 1$, and $\Lambda^{k+1}\succeq 0$. □

We next show the convergence property of Algorithm 1.

Algorithm 3: The HSA Based on DCA.

1: **Input:** Initial temperature T_0, constant β and initial state X_{curr}.
2: **Output:** A matrix X.
3: **Loop: for** $k=2$ **to** MAXiter
- $X_{\text{new}} = X_{\text{curr}} + Z$, where $Z_{i,j} \overset{i.i.d.}{\sim} \mathcal{N}(0, T+\beta)$.
- Get the output DCA(X_{new}) from DCA in Algorithm 1 by X_{new} taken as the initialization.
- $X_{\text{curr}} :=$ $\begin{cases} \text{DCA}(X_{\text{new}}) & F(\text{DCA}(X_{\text{new}})) < F(X_{\text{curr}}) - T\ln(\alpha) \\ \mathbf{0} & \text{otherwise,} \end{cases}$
  where $\alpha \sim \mathcal{U}(0,1)$ as some random number in (0,1).
  $T := T_0/\log(k)$.

4: $X = X_{\text{curr}}$.

Theorem V.2: Assume that $\{X^k\}_{k\ge 1}$ is a sequence generated by Algorithm 1. We have

1) $\{X^k\}_{k\ge1}$ is a bounded sequence;
2) $\lim_{k\to\infty}\|X^{k+1}-X^k\|_F = 0$;
3) Assume that $\widetilde{X} = \widetilde{X}_1 + i\widetilde{X}_2$ is an accumulation point of $\{X^k\}_{k\ge1}$. Then $\widetilde{X}$ satisfies:

1) Stationary condition:

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix} \Big/ \left\|\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\|_F\right) + \mu\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix} + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right) - \mathbf{b}\right) - \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix} = 0; \quad (53)$$

2) Complementary slackness condition:

$$\left\langle \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}, \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\rangle = 0, \quad (54)$$

for some $\begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix} \in \mathbb{H}_+^{d\times d}$ and

$$\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix} \in \partial\left(\left\|\begin{bmatrix}\text{vec}(\widetilde{X}_1)\\ \text{vec}(\widetilde{X}_2)\end{bmatrix}\right\|_{1,2}\right), \quad (55)$$

where $\partial(\|\begin{bmatrix}\text{vec}(\widetilde{X}_1)\\ \text{vec}(\widetilde{X}_2)\end{bmatrix}\|_{1,2})$ is given in (35).

Remark V.3: The theorem above only provides the local convergence of DCA algorithm. However, the numerical experiments in Section VI show that DCA algorithm behaves better than other state-of-art algorithms. We would like to mention that one can employ Hybrid Simulated Annealing (HSA) (see [10], [31]) to traverse local minima generated from DCA to some global solution. More details can be seen in Algorithm 3.

3) According to Theorem 4.1 in [31], HSA algorithm can converge to the global minimum in probability.

Proof: (1) The definition of F implies $\mu\|X^{k+1}\|_1 \le F(X^{k+1})$ and hence $\|X^{k+1}\|_1 \le F(X^{k+1})/\mu \le F(X^0)/\mu$ for $k\ge1$. Here we use Lemma V.1, i.e., $\{F(X^k)\}_{k\ge1}$ is monotonically decreasing. Hence, $\{X^k\}_{k\ge1}$ is a bounded sequence.

(2) We first consider the case where $X^1 = \mathbf{0}$. A simple calculation shows that $X^k = \mathbf{0}$ for $k\ge 2$ provided that $X^1=\mathbf{0}$, and we arrive at the conclusion immediately. So, we next just consider the case on $X^1 \ne \mathbf{0}$. Taking $k=0$ in (52), we obtain that

$$F(X^0) - F(X^1) = F(\mathbf{0}) - F(X^1) = \frac{1}{2}\|\mathcal{A}(X^1)\|_2^2 + \lambda\|X^1\|_F \ge \lambda\|X^1\|_F > 0$$

as $Y^k = \mathbf{0}$. It implies $F(X^k) \le F(X^1) < F(\mathbf{0})$ for any $k\ge2$. Hence, we obtain that $X^k \ne 0$ for all $k\ge1$. By (52), we obtain that

$$F(X^k) - F(X^{k+1}) \ge \frac{1}{2}\|\mathcal{A}(X^{k+1}-X^k)\|_2^2 + \lambda(\|X^{k+1}\|_F - \langle X^{k+1}, Y^k\rangle).$$

Noting that $\{F(X^k)\}_{k\ge1}$ is a convergent sequence and $\|X^{k+1}\|_F - \langle X^{k+1}, Y^k\rangle \ge 0$, we have

$$\lim_{k\to\infty}\|\mathcal{A}(X^k - X^{k+1})\|_2 = 0 \quad (56)$$

and

$$\lim_{k\to\infty}(\|X^{k+1}\|_F - \langle X^{k+1}, Y^k\rangle) = \lim_{k\to\infty}\left(\|X^{k+1}\|_F - \left\langle X^{k+1}, \frac{X^k}{\|X^k\|_F}\right\rangle\right) = 0. \quad (57)$$

The following argument is similar with that in Proposition 3.1 (b) in [31]. We put it here for completeness. Set $c_k := \frac{\langle X^k, X^{k+1}\rangle}{\|X^k\|_F^2}$ and $E^k := X^{k+1} - c_k X^k$. It suffices to prove that $E^k \to \mathbf{0}$ and $c_k \to 1$. According to (57) and boundness of $\{X^k\}_{k\ge1}$, we have

$$\|E^k\|_F^2 = \|X^{k+1}\|_F^2 - \frac{\langle X^k, X^{k+1}\rangle^2}{\|X^k\|_F^2} = \left(\|X^{k+1}\|_F - \frac{\langle X^k, X^{k+1}\rangle}{\|X^k\|_F^2}\right)\left(\|X^{k+1}\|_F + \frac{\langle X^k, X^{k+1}\rangle}{\|X^k\|_F^2}\right) \to 0,$$

Then we have

$$0 = \lim_{k\to\infty}\|\mathcal{A}(X^k - X^{k+1})\|_2 = \lim_{k\to\infty}\|\mathcal{A}((c_k-1)X^k - E^k)\|_2 = \lim_{k\to\infty}|c_k - 1|\|\mathcal{A}(X^k)\|_2.$$

If $\lim_{k\to\infty} c_k \ne 1$, then there exists a subsequence $\{X^{k_j}\}$ such that $\|\mathcal{A}(X^{k_j})\|_2 \to 0$. Therefore, we can obtain that

$$\lim_{k_j\to\infty} F(X^{k_j}) \ge \lim_{k_j\to\infty}\frac{1}{2}\|\mathcal{A}(X^{k_j}) - \mathbf{b}\|_2^2 = \frac{1}{2}\|\mathbf{b}\|_2^2 = F(X^0),$$

which leads to a contradiction to the fact that

$$F(X^{k_j}) \le F(X^1) < F(X^0).$$

Thus we can get $c_k \to 1$, $E^k \to \mathbf{0}$, and thus $X^{k+1} - X^k \to \mathbf{0}$, when $k\to\infty$.

(3) Assume that $\{X^{k_j}\}_{j\geq 1} \subset \{X^k\}_{k\geq 1}$ is a subsequence satisfying $\lim_{j\to\infty} X^{k_j} = \widetilde{X} = \widetilde{X}_1 + i\widetilde{X}_2 \neq \mathbf{0}$. For simplicity, we abuse the notation and denote $\{X^{k_j}\}$ as $\{X^j\}$. Replacing k by $j-1$ in (48) and (49), we have

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}Y_1^{j-1}\\ Y_2^{j-1}\end{bmatrix}\right) + \mu\begin{bmatrix}G_1^j\\ G_2^j\end{bmatrix} + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^j\\ X_2^j\end{bmatrix}\right) - \mathbf{b}\right) = \begin{bmatrix}\Lambda_1^j\\ \Lambda_2^j\end{bmatrix}, \tag{58}$$

and

$$\left\langle \begin{bmatrix}\Lambda_1^j\\ \Lambda_2^j\end{bmatrix}, \begin{bmatrix}X_1^j\\ X_2^j\end{bmatrix}\right\rangle = 0, \tag{59}$$

for some $\Lambda^j = \Lambda_1^j + i\Lambda_2^j$ with $\left[\begin{smallmatrix}\Lambda_1^j\\ \Lambda_2^j\end{smallmatrix}\right] \in \mathbb{H}_+^{d\times d}$, and $G^j = G_1^j + iG_2^j$ with

$$\begin{bmatrix}G_1^j\\ G_2^j\end{bmatrix} \in \partial\left(\left\|\begin{bmatrix}\operatorname{vec}(X_1^j)\\ \operatorname{vec}(X_2^j)\end{bmatrix}\right\|_{1,2}\right). \tag{60}$$

Note that (58) is equivalent to

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}Y_1^{j-1}\\ Y_2^{j-1}\end{bmatrix}\right) + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^j\\ X_2^j\end{bmatrix}\right) - \mathbf{b}\right) = \begin{bmatrix}\Lambda_1^j\\ \Lambda_2^j\end{bmatrix} - \mu\begin{bmatrix}G_1^j\\ G_2^j\end{bmatrix}. \tag{61}$$

Noting that

$$\lim_{j\to\infty}\begin{bmatrix}Y_1^{j-1}\\ Y_2^{j-1}\end{bmatrix} = \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix} \Big/ \left\|\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\|_F,$$

we obtain that the left hand side of (61) converges to

$$\lim_{j\to\infty}\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}Y_1^{j-1}\\ Y_2^{j-1}\end{bmatrix}\right) + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}X_1^j\\ X_2^j\end{bmatrix}\right) - \mathbf{b}\right) = \lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix} \Big/ \left\|\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\|_F\right) + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right) - \mathbf{b}\right). \tag{62}$$

For convenience, we set

$$P^j := \begin{bmatrix}\Lambda_1^j\\ \Lambda_2^j\end{bmatrix} \quad\text{and}\quad Q^j := -\mu\begin{bmatrix}G_1^j\\ G_2^j\end{bmatrix}.$$

According to (60) and Proposition IV.3, we have $\|G_1^j\|_\infty \leq 1$ and $\|G_2^j\|_\infty \leq 1$. Combining (61) and the boundedness of $\{X^j\}_{j\geq 1}$, we obtain that $\{P^j\}_{j\geq 1}$ and $\{Q^j\}_{j\geq 1}$ are also bounded sequences, which can belong to some compact sets $S \subset \mathbb{H}_+^{d\times d}$ and T, respectively.

We assume that $\{j_l\}_{l\geq 1}$ is a subsequence of $\{j\}_{j\geq 1}$ such that $\lim_{l\to\infty} P_{j_l} = \widetilde{P}$ and $\lim_{l\to\infty} Q_{j_l} = \widetilde{Q}$ for some $\widetilde{P} \in S, \widetilde{Q} \in T$. More concretely, we have

$$\lim_{l\to\infty} P^{j_l} = \lim_{l\to\infty}\begin{bmatrix}\Lambda_1^{j_l}\\ \Lambda_2^{j_l}\end{bmatrix} = \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix},$$

and

$$\lim_{l\to\infty} Q^{j_l} = \lim_{l\to\infty} -\mu\begin{bmatrix}G_1^{j_l}\\ G_2^{j_l}\end{bmatrix} = -\mu\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix}$$

for some

$$\begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix} \in S \subset \mathbb{H}_+^{d\times d}.$$

According to (61), we have

$$\lambda\left(\begin{bmatrix}\mathbf{I}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix} \Big/ \left\|\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\|_F\right) + \widetilde{\mathcal{A}}^*\left(\widetilde{\mathcal{A}}\left(\begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right) - \mathbf{b}\right) = \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix} - \mu\begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix},$$

which implies the stationary condition (53). The complementary slackness condition (54) is obtained by

$$\left\langle \begin{bmatrix}\widetilde{\Lambda}_1\\ \widetilde{\Lambda}_2\end{bmatrix}, \begin{bmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{bmatrix}\right\rangle = \lim_{l\to\infty}\left\langle \begin{bmatrix}\Lambda_1^{j_l}\\ \Lambda_2^{j_l}\end{bmatrix}, \begin{bmatrix}X_1^{j_l}\\ X_2^{j_l}\end{bmatrix}\right\rangle = 0.$$

Here, we use (59).

We remain to prove (55). For sufficiently large j_l, we have $\operatorname{supp}(\widetilde{X}) \subset \operatorname{supp}(X^{j_l})$. If $(i_1, i_2) \in \operatorname{supp}(\widetilde{X})$, then

$$\lim_{l\to\infty}([G_1^{j_l}]_{i_1,i_2}, [G_2^{j_l}]_{i_1,i_2}) = \frac{([\widetilde{X}_1]_{i_1,i_2}, [\widetilde{X}_2]_{i_1,i_2})}{\sqrt{[\widetilde{X}_1]_{i_1,i_2}^2 + [\widetilde{X}_2]_{i_1,i_2}^2}}.$$

If $(i_1, i_2) \notin \operatorname{supp}(\widetilde{X})$, we have

$$([G_1^{j_l}]_{i_1,i_2})^2 + ([G_2^{j_l}]_{i_1,i_2})^2 \leq 1$$

and hence

$$([\widetilde{G}_1]_{i_1,i_2})^2 + ([\widetilde{G}_2]_{i_1,i_2})^2 \leq 1.$$

Thus

$$\lim_{l\to\infty}\begin{bmatrix}G_1^{j_l}\\ G_2^{j_l}\end{bmatrix} = \begin{bmatrix}\widetilde{G}_1\\ \widetilde{G}_2\end{bmatrix} \in \partial\left(\left\|\begin{bmatrix}\operatorname{vec}(\widetilde{X}_1)\\ \operatorname{vec}(\widetilde{X}_2)\end{bmatrix}\right\|_{1,2}\right),$$

which leads to (55). ■

VI. Numerical Experiments

The purpose of numerical experiments is to compare the performance of (8) with that of SPARTA[1] [25], of SWF[2] [32], of SPRSF [17] and of CoPRAM[3] [13]. We choose the parameters of those algorithms as those in the corresponding codes. In Algorithm 1, we estimate signal $\mathbf{x}^\#$ by extracting the largest

[1]The codes of SPARTA are at: https://gangwg.github.io/SPARTA/

[2]The codes of SWF are at: https://github.com/Ziyang1992/Sparse-Wirtinger-flow.git

[3]The codes of CoPRAM are at: https://github.com/GauriJagatap/model-copram

rank-1 component of $X^{\#}$. If the eigenvalue decomposition of $X^{\#}$ is taken as $X^{\#} = \sum_{k=1}^{d} \lambda_k \mathbf{u}_k \mathbf{u}_k^*$ with $\lambda_1 \geq \cdots \geq \lambda_n$, we set $\mathbf{x}^{\#} = \sqrt{\lambda_1}\mathbf{u}_1$.

In this section, we use the relative error

$$\text{relative error} := \frac{d_r(\mathbf{x}^{\#}, \mathbf{x}_0)}{\|\mathbf{x}_0\|_2},$$

where $d_r(\mathbf{x}^{\#}, \mathbf{x}_0) := \min \|\mathbf{x}^{\#} \pm \mathbf{x}_0\|_2$ for the real case and $d_r(\mathbf{x}^{\#}, \mathbf{x}_0) := \min_{\theta \in [0,2\pi)} \|\exp(-i\theta)\mathbf{x}^{\#} - \mathbf{x}_0\|_2$ for the complex case. In our numerical experiments, we focus on two kinds of measurements, i.e., the Gaussian model and the coded diffraction pattern (CDP) model: (i) the Gaussian model: the sampling vectors $\mathbf{a}_j, j = 1, \ldots, m$, are Gaussian random vector, i.e., $\mathbf{a}_j \sim \mathcal{N}(0, \mathbf{I}_d)$ for real case and $\mathbf{a}_j \sim \mathcal{N}(0, \frac{1}{2}\mathbf{I}_d) + i\mathcal{N}(0, \frac{1}{2}\mathbf{I}_d)$ for complex case; (ii) the CDP model: the sampling vectors $\mathbf{a}_{p,q} := D_p \mathbf{f}_q$, where $\mathbf{f}_q^*$ is the q-th row of the $d \times d$ discrete Fourier Transform matrix, and D_p is a diagonal matrix with entries independently sampled from a distribution g. Here we take the octanary pattern $g = g_1 g_2$ as in [6], where g_1 and g_2 are independent with distribution

$$g_1 = \begin{cases} 1 & \text{with prob. } 1/4 \\ -1 & \text{with prob. } 1/4 \\ i & \text{with prob. } 1/4 \\ -i & \text{with prob. } 1/4 \end{cases} \quad g_2 = \begin{cases} \frac{\sqrt{2}}{2} & \text{with prob. } 4/5 \\ \sqrt{3} & \text{with prob. } 1/5 \end{cases}. \tag{63}$$

For each fixed k, the support of a k-sparse signal $\mathbf{x}_0$ is drawn from the uniform distribution over the set of all subsets of $[1, m] \cap \mathbb{Z}$ of size k. The non-zero entries of the real (resp. complex) k-sparse signal $\mathbf{x}_0$ have Gaussian distribution $\mathcal{N}(0, 1)$ (resp. $\mathcal{N}(0, 1) + i\mathcal{N}(0, 1)$). In order to reduce dimension effect, we normalize $\mathbf{x}_0$ into $\|\mathbf{x}_0\|_2 = 1$. All experiments are carried out on Matlab 2017 with a 3.7 GHz Intel Core i7-8700 K and 64 GB memory. The code is available at https://github.com/xiayu03235/SparsePhaseliftOff.

Example VI.1: The aim of this numerical experiment is to test the success rate of Algorithm 1 against the measurement number in both Gaussian model and CDP model. In this example, we take $k = 5$ and $d = 50$. In Gaussian model, the ratio between m and d is varied from 0.1 to 4, with stepsize 0.1; In CDP model, L is varied from 2 to 8 with stepsize 1. In both cases, we choose $\mu = 0.001$ and $\lambda = \frac{\mu k}{\sqrt{2}-1}$ in Algorithm 1. We classify a recovery as a success if the relative error is less than 10^{-3}. For each fixed measurement number, we repeat the experiments for 40 trails and and calculate the success rate.

Figure 1 depicts the empirical probability of successful recovery against the measurement number. The numerical results show that Algorithm 1 outperforms other algorithms. Besides, we can see that Algorithm 1 has good performance for both kinds of measurement, while other algorithms only behave well for the Gaussian measurement.

Example VI.2: In this example, we test the success rate of Algorithm 1 against the sparsity level k. We take $d = 50$ and $m = 2\,d$ for Gaussian case, and $d = 50$, $L = 5$ for CDP case. The parameters in Algorithm 1 are taken as $\mu = 0.001$ and $\lambda = \frac{\mu k}{\sqrt{2}-1}$. Figure 2 depicts the numerical results. It shows that Algorithm 1 is superior to the SPRSF, SWF, SPARTA and CoPRAM for both the Gaussian model and the CDP model. Furthermore, we can see that Algorithm 1 can make good performance even under large level of sparsity.

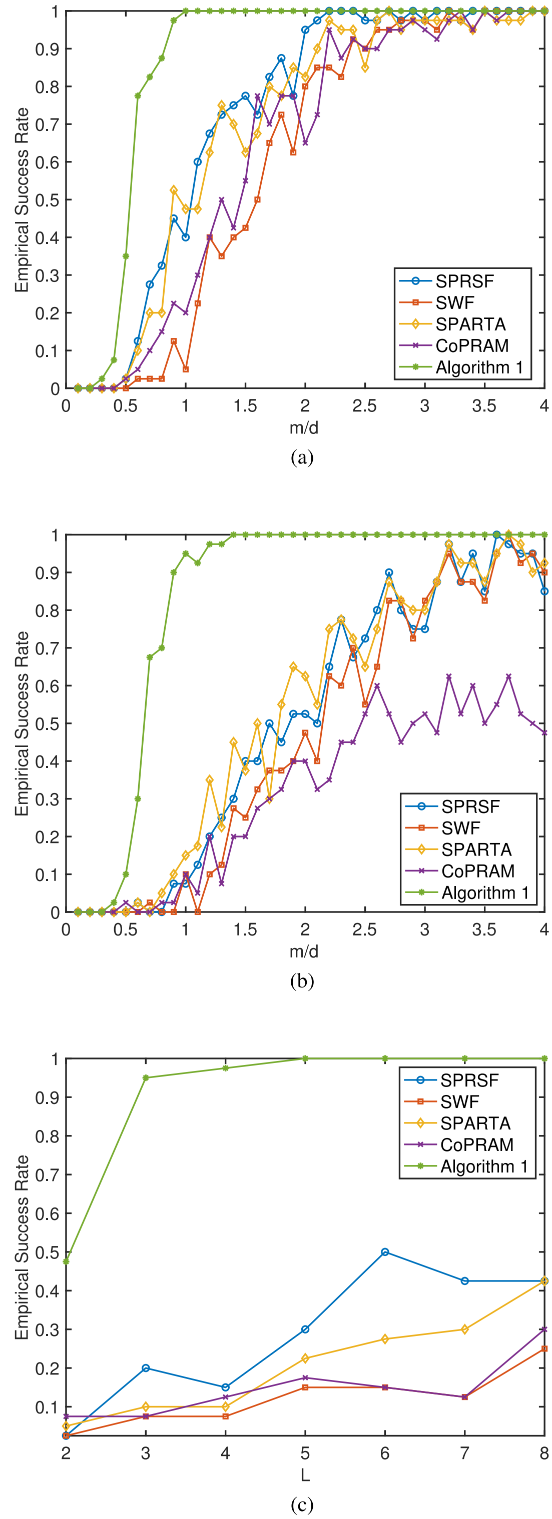

Fig. 1. Comparison of different algorithms for fixed $k = 5$ with different measurements: (a) Noiseless real-valued Gaussian model; (b) Noiseless complex-valued Gaussian model; (c) Noiseless coded diffraction model.

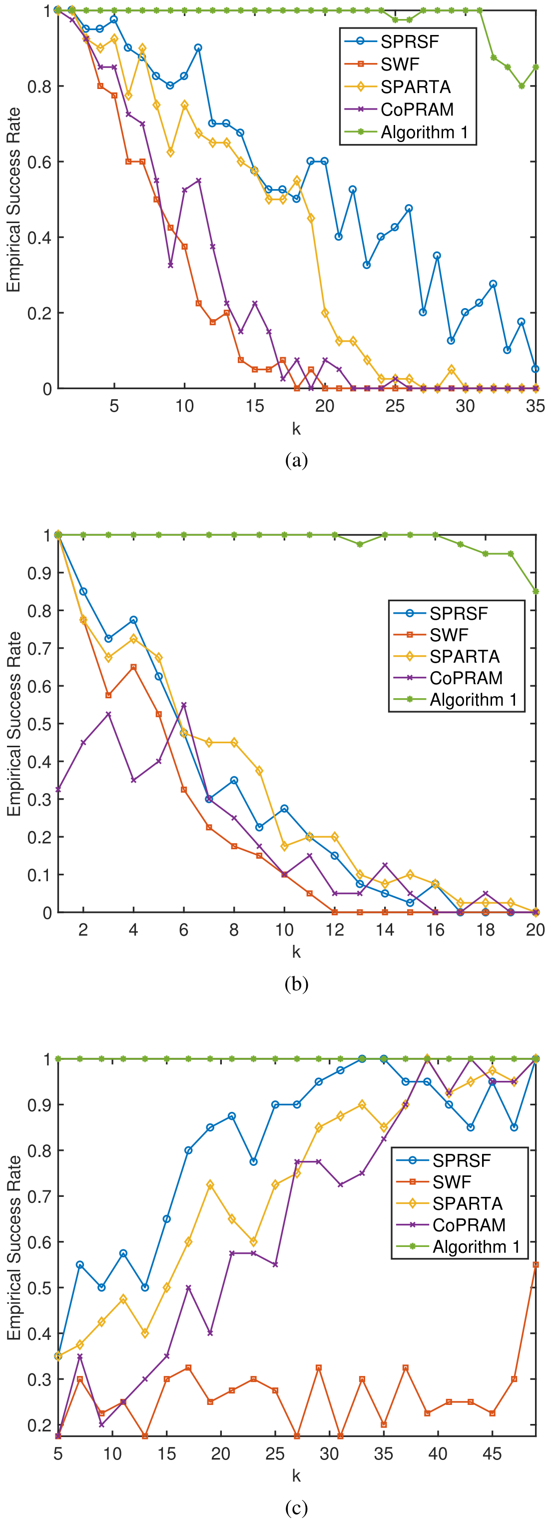

Fig. 2. Comparison of different algorithms for different sparsity level k: (a) Noiseless real-valued Gaussian model; (b) Noiseless complex-valued Gaussian model; (c) Noiseless coded diffraction model.

Example VI.3: In this example, we test the robustness of Algorithm 1. We take $d = 50$, $m = 2\,d$ and $k = 5$ for Gaussian case, and $d = 50$, $L = 5$ for CDP case. The white Gaussian noise is followed by MATLAB function **awgn($\mathcal{A}(\mathbf{x}_0)$,snr)**, i.e., $b_j = |\langle \mathbf{a}_j, \mathbf{x}_0\rangle|^2 + w_j, j = 1, \ldots, m$

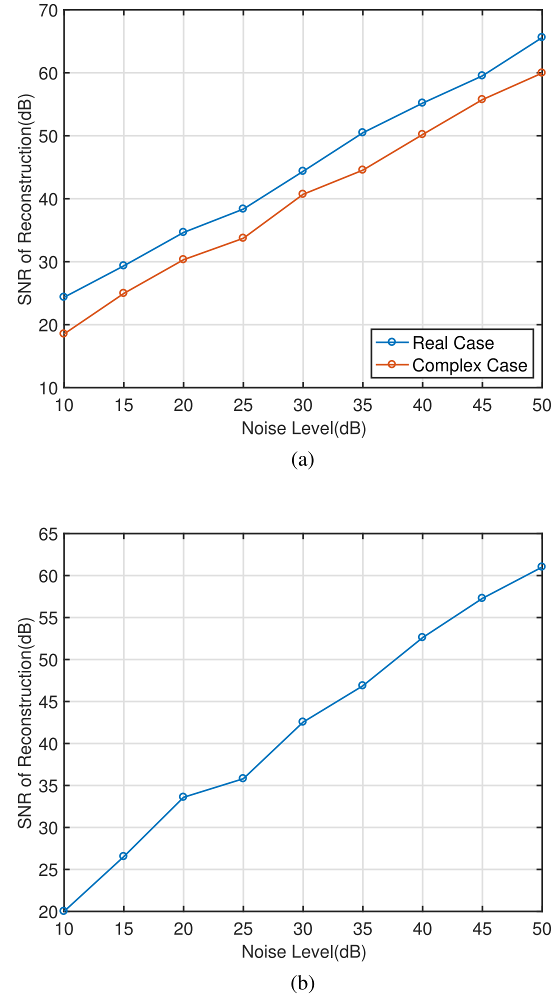

Fig. 3. SNR of signal recovery v.s. noise level in measurements when $k = 5$, x-axis is the noise level varying from 10db to 50db, y-axis is the reconstruction error in db as $-20\log_{10}$(relative error): (a) Noisy Gaussian model; (b) Noisy coded diffraction model.

with $\mathbf{w} \sim \sqrt{\frac{\|\mathcal{A}(\mathbf{x}_0)\|_2^2/m}{10^{snr/10}}}\mathcal{N}(0, \mathbf{I}_m)$. Since other algorithms do not make 100% recovery under this setting, we only show the robustness performance on Algorithm 1. The SNR value varies from 10 dB to 50 dB, with step-size 5 dB. The SNR in each noise level is averaged over 20 independent trials. According to Theorem I.2, we choose $\mu = \max\{0.5\|\mathbf{w}\|_2, 0.001\}$ and $\lambda = \frac{\mu k}{\sqrt{2}-1}$. We compute the signal-to noise ratio of reconstruction in dB as $-20\log_{10}$(relative error). In Figure 3 , it shows that Algorithm 1 yields robust recovery with respect to different noise level.

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